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MATHEMATICAL

SOLUTION BOOK

CONTAINING

Systematic Solutions

TO MANY OF THE

Most Difficult Problems.

TAKEN FROM THE LEADING AUTHORS ON ARITHMETIC AND ALGEBRA, MANY PROBLEMS AND SOLUTIONS FROM GEOMETRY, TRIGONOMETRY AND CALCULUS, MANY PROBLEMS AND SOLUTIONS FROM THE LEADING MATHEMATICAL JOURNALS OF THE UNITED STATES, AND MANY ORIGINAL PROBLEMS AND SOLUTIONS,

WITH

NOTES AND EXPLANATIONS,

BY

B. F. FINKEL,

Member of the New York Mathematical Society, and Professor of Mathematics in the Kidder Institute.

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DEDICATED

TO MY FRIEND,

PROF. H. S. LEHR, A. M.,

PRESIDENT OF

THE OHIO NORMAL UNIVERSITY.

PREFACE.

This work is the outgrowth of eight years' experience in teaching in the Public Schools, during which time I have observed that a work presenting a systematic treatment of solutions to problems would be serviceable to both teachers and pupils.

It is not intended to serve as a key to any work on mathematics; but the object of its appearance is to present, for use in the schoolroom, such an accurate and logical method of solving problems as will best awaken the latent energies of pupils, and teach them to be original investigators in the various branches of science.

It will not be denied by any intelligent educator that the so called "Short Cuts" and "Lightning Methods" are positively injurious to beginners in mathematics. All the "whys" are cut out by these methods and the student robbed of the very object for which he is studying mathematics; *viz.*, the development of the reasoning faculty and the power to express his thoughts in a forcible and logical manner. By pursuing these methods, mathematics is made a mere memory drill and when the memory fails, all is lost; whereas, it should be presented in such a way as to develop the memory, the imagination, and the reasoning faculty. By following out the method pursued in this book, the mind will be strengthened in these three powers, besides a taste for neatness and a love of the beautiful will be cultivated.

Any one who can write out systematic solutions to problems can resort to "Short Cuts" at pleasure; but, on the other hand, let a student who has done all his work in mathematics by formulæ, "Short Cuts," and "Lightning Methods" attempt to write

out a systematic solution—one in which the work explains itself—and he will soon convince you of his inability to express his thoughts in a logical manner. These so-called "Short Cuts" should not be used at all, in the schoolroom. After pupils and students have been drilled on the systematic method of solving problems, they will be able to solve more problems by short methods than they could, by having been instructed in all the "Short Cuts" and "Lightning Methods" extant.

It can not be denied that more time is given to, and more time wasted in the study of arithmetic in the public schools than in any other branch of study; and yet, as a rule, no better results are obtained in this branch than in any other. The reason of this, to my mind, is apparent. Pupils are allowed to combine the numbers in such a way as "to get the answer" and that is all that is required. They are not required to tell why they do this, or why they do that, but, "did you get the answer?" is the question. The art of "ciphering" is thus developed at the expense of the reasoning faculty.

The method of solving problems pursued in this book is often called the "Step Method." But we might, with equal propriety, call any orderly manner of doing any thing, the "Step Method." There are only two methods of solving problems—a right method and a wrong method. That is the right method which takes up, in logical order, link by link, the chain of reasoning and arrives at the correct result. Any other method is wrong and hurtful when pursued by those who are beginners in mathematics.

One solution, thoroughly analyzed and criticised by a class, is worth more than a dozen solutions the difficulties of which are seen through a cloud of obscurities.

This book can be used to a great advantage in the classroom—the problems at the end of each chapter affording ample exercise for supplementary work.

Many of the Formulæ in Mensuration have been obtained by the aid of the Calculus, the operation alone being indicated. This

feature of the work will not detract any from its merits for those persons who do not understand the Calculus ; for those who do understand the Calculus it will afford an excellent drill to work out all the steps taken in obtaining the formulæ. Many of the formulæ can be obtained by elementary geometry and algebra. But the Calculus has been used for the sake of presenting the beauty and accuracy of that powerful instrument of mathematics.

In cases in which the formulæ lead to series, as in the case of the circumference of the ellipse, the rule is given for a near approximation.

It has been the aim to give a solution to every problem presenting any thing peculiar, and to those which go the rounds of the country. Any which have been omitted will receive space in future editions of this work. The limits of this work have compelled me to omit much curious and valuable matter in Higher Mathematics.

I have taken some problems and solutions from the *School Visitor*, published by John S. Royer ; the *Mathematical Magazine*, and the *Mathematical Visitor*, published by Artemas Martin, A. M., Ph. D., LL. D.; and the *Mathematical Messenger*, published by G. H. Harvill, by the kind permission of these distinguished gentlemen.

It remains to acknowledge my indebtedness to Prof. William Hoover, A. M., Ph. D., of the Department of Mathematics and Astronomy in the Ohio University at Athens, for critically reading the manuscript of the part treating on Mensuration.

Hoping that the work will, in a measure, meet the object for which it was written, I respectfully submit it to the use of my fellow teachers and co-laborers in the field of mathematics.

Any correction or suggestion will be thankfully received by communicating the same to me immediately.

THE AUTHOR.

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CHAPTER I.

DEFINITIONS.

1. *Mathematics* (*μαθηματική*, science) is that science which treats of quantity.

- MATHEMATICS.
  - I. Pure.
    - (1.) Arithmetic.
    - (2.) Algebra...
      - 1. Calculus...........
        - a. Differential.
        - b. Integral.
        - c. Calculus of Variations.
      - 2. Quaternions.
    - (3.) Geometry..
      - 1. Platonic Geometry..
        - a. Pure Geometry.
        - b. Conic Sections.
        - c. Trigonometry..
          - 1. Plane Trigon'y.
          - 2. Analytical Trig.
          - 3. Spherical "
      - 2. Analytical Geometry.
      - 3. Descriptive Geometry.
  - II. Applied.
    - (1.) Mensuration.
    - (2.) Surveying.
    - (3.) Navigation.
    - (4.) Mechanics.
    - (5.) Astronomy.
    - (6.) Optics.
    - (7.) Gunnery.
    - (8.) &c., &c.

2. *Pure Mathematics* treats of magnitude or quantity without relation to matter.

3. *Applied Mathematics* treats of magnitude as subsisting in material bodies.

4. *Arithmetic* (*ἀριθμητική*, from *ἀριθμός*, a number) is the science of numbers and the art of computing by them.

5. *Algebra* (*Ar. al*, the, and *geber*, philosopher) is that method of mathematical computation in which letters and other symbols are employed.

6. *Geometry* (*γεωμετρία*, from *γεωμετρεῖν*, to measure land, from *γέα*, *γῆ*, the earth, and *μετρεῖν*, to measure) is the science of position and extension.

7. *Calculus* (*Calculus*, a pebble) is that branch of mathematics which commands by one general method, the most difficult problems of geometry and physics.

8. ***Differential Calculus*** is that branch of Calculus which investigates mathematical questions by measuring the relation of certain infinitely small quantities called *differentials.*

9. ***Integral Calculus*** is that branch of Calculus which determines the functions from which a given differential has been derived.

10. ***Calculus of Variations*** is that branch of calculus in which the laws of dependence which bind the variable quantities together are themselves subject to change.

11. ***Quaternions*** (*quaternis*, from *quaterni* four each, from *quator*, four) is that branch of algebra which treats of the relations of magnitude and position of lines or bodies in space by means of the quotient of two direct lines in space, considered as depending on a system of four geometrical elements, and as expressed by an algebraic symbol of quadrinominal form.

12. ***Platonic Geometry*** is that branch of geometry in which the argument is carried forward by a direct inspection of the figures themselves, delineated before the eye, or held in the imagination.

13. ***Pure Geometry*** is that branch of Platonic geometry in which the argument may be practically tested by the aid of the compass and the square only.

14. ***Conic Sections*** is that branch of Platonic geometry which treats of the curved lines formed by the intersection of a cone and a plane.

15. ***Trigonometry*** (*τρίγωνον*, triangle, *μετρον*, measure) is that branch of Platonic geometry which treats of the relations of the angles and sides of triangles.

16. ***Plane Trigonometry*** is that branch of trigonometry which treats of the relations of the angles and sides of plane triangles.

17. ***Analytical Trigonometry*** is that branch of trigonometry which treats of the general properties and relations of trigonometrical functions.

18. ***Spherical Trigonometry*** is that branch of trigonometry which treats of the solution of spherical triangles.

19. ***Analytical Geometry*** is that branch of geometry in which the properties and relations of lines and surfaces are investigated by the aid of algebraic analysis.

20. ***Descriptive Geometry*** is that branch of geometry which seeks the graphic solution of geometrical problems by means of projections upon auxiliary planes.

21. ***Mensuration*** is that branch of applied mathematics which treats of the measurment of geometrical magnitudes.

22. ***Surveying*** is that branch of applied mathematics which treats of the art of determining and representing distances, areas, and the relative position of points upon the earth's surface.

23. ***Navigation*** is that branch of applied mathematics which treats of the art of conducting ships from one place to another.

24. ***Mechanics*** is that branch of applied mathematics which treats of the laws of equilibrium and motion.

25. ***Astronomy*** (*αστρονομια,* from *αστρον*, star and *νομος* law) is that branch of applied mathematics in which mechanical principles are used to explain astronomical facts.

26. ***Optics*** (*οπτίκή,* from *οπίς,* sight,) is that branch of applied mathematics which treats of the laws of light.

27. ***Gunnery*** is that branch of applied mathematics which treats of the theory of projectiles.

28. ***A Proposition*** is a statement of something proposed to be done.

29. Prop't'n.
- 1. Demonstrable.
  - *a.* Theorem.
    - 1. Lemma.
    - 2. Corollary.
  - *b.* Problem.
- 2. Indemonstrable.
  - *a.* Axiom.
  - *b.* Postulate.

30. ***A Demonstrable Proposition*** is one that can be proved by the aid of reason.

31. ***A Theorem*** is a truth requiring a proof.

32. ***A Lemma*** is a theorem demonstrated for the purpose of using it in the demonstration of another theorem.

33. ***A Corollary*** is a subordinate theorem, the truth of which is made evident in the course of the demonstration of a more general theorem.

34. ***A Problem*** is a question proposed for solution.

35. ***An Indemonstrable Proposition*** can not be proved by any manner of reasoning.

36. ***An Axiom*** is a self-evident truth.

37. ***A Postulate*** is a proposition which states that something can be done, and which is so evidently true as to require no process of reasoning to show that it is possible to be done.

38. ***A Demonstration*** is the process of reasoning, proving the truth of a proposition.

39. ***A Solution*** of a problem is an expressed statement showing clearly how the result is obtained.

40. ***An Operation*** is a process of finding, from given quantities, others that are known, by simply illustrating the solution.

41. ***A Rule*** is a general direction for solving all problems of a particular kind.

42. ***A Formula*** is the expression of a general rule or principle in algebraic language.

43. ***A Scholium*** is a remark made at the close of a discussion, and designed to call attention to some particular feature or features of it.

CHAPTER II.

NUMERATION AND NOTATION.

1. ***Numeration*** is the art of reading numbers.

2. There are two methods of numeration ; the *French* and the *English.*

3. The *French* method is that in general use. In this method, we begin at the right hand and divide the number into periods of three figures each, and give a distinct name to each period.

4. The *English* method is that used in Great Britain and the British provinces. In this method, we divide the number (if it consists of more than six figures) into periods of six figures each, and give a distinct name to each period. The following number illustrates the two methods ; the upper division showing how the number is read by the English method, and the lower division showing how it is read by the French method.

4th period, Trillions.	3d period, Billions.		2d period, Millions.		1st period. Units.	
845	678	904	325	147	434	913
7th period, Quintillions.	6th period, Quadrillions.	5th period, Trillions.	4th period, Billions.	3d period, Millions.	2d period, Thousands.	1st period, Units.

5. The number expressed in words by the English method, reads thus:

Eight hundred forty-five trillion, six hundred seventy-eight thousand nine hundred four billion, three hundred twenty-five thousand one hundred forty-seven million, four hundred thirty-four thousand nine hundred thirteen.

Remark.—Use the conjunction *and*, only in passing over the decimal point. It is incorrect to read 456,734 four hundred and fifty-six thousand, seven hundred and thirty-four. Omit the *ands* and the number will be correctly expressed in words.

6. The following are the names of the Periods, according to the common, or French method:

First	Period,	Units.	Sixth	Period,	Quadrillions.
Second	"	Thousands.	Seventh	"	Quintillions.
Third	"	Millions.	Eighth	"	Sextillions.
Fourth	"	Billions.	Ninth	"	Septillions.
Fifth	"	Trillions,	Tenth	"	Octillion.

Other periods in order are, Nonillions, Decillions, Undecillions, Duodecillions, Tredecilions, Quatuordecillions Quindecillions, Sexdecillions, Septendecillions, Octodecillions, Novendecillions, Vigintillions, Primo-Vigintillions, Secundo-vigintillions, Tertio-vigintillions, Quarto-vigintillions, Quinto-vigintillions, Sexto-vigintillions, Septo-vigintillions, Octo-vigintillions, Nonovigintillions, Trigillions; Primo-Trigillions, Secundo-Trigillions, and so on to Quadragillions; Primo-quadragillions, Secundoquadragillions, and so on to Quinquagillions; Primo-quinquagillions, Secundo-quinquagillions, and so on to Sexagillions, Primo-sexagillions, Secundo-sexagillions. and so on to Septuagillions; Primo-septuagillions, Secundo-septuagillions, and so on to Octogillions; Primo-octogillions, Secundo-octogillions, and so on to Nonogillions; Primo-nonogillions, Secundo-nonogillions, and so to Centillions.

7. ***Notation*** is the art of writing numbers.

There are three methods of expressing numbers; by words, by letters, called the *Roman* method, and by figures, called the *Arabic* method.

8. The *Roman Notation*, so called from its having originated with the ancient Romans, uses seven capital letters to express numbers; viz., I, V, X, L, C, D, M.

9. The *Arabic Notation*, so called from its having been made known through the Arabs, uses ten characters to express numbers; viz., 1, 2, 3, 4, 5, 6, 7, 8, 9, 0.

EXAMPLES.

1. Write three hundred seventy quadrillion, one hundred one thousand one hundred thirty-four trillion, seven hundred eighty-

nine thousand six hundred thirty-two billion, two hundred ninety-eight thousand seven hundred sixty-five million, four hundred thirty-seven thousand one hundred fifty-six.

2. Read by the *English* method, 789432781023457893289O3-24678.

3. Write three thousand one hundred forty-one quintillion, five hundred ninety-two billion six hundred fifty-three million five hundred eighty- nine thousand seven hundred ninety-three quadrillion, two hundred thirty-eight billion four hundred sixty-two million six hundred forty-three thousand three hundred eighty-three trillion, two hundred seventy-nine billion five hundred two million, eight hundred eighty-four thousand one hundred ninety-seven.

4. Read 141421356237309504880168872420969807856971437-89132.

5. Is a billion, a million million? Explain.

6. Write 19 billion billion billion.

7. Write 19 trillion billion million million.

8. Write 19 hundred 56 thousand.

9. Write 457 thousand 341 million.

10. Write 19 trillion trillion billion billion million million.

CHAPTER III.

ADDITION.

1. ***Addition*** is the process of uniting two or more numbers of the same kind into one sum or amount.

2. Add the following, beginning at the right, and prove the result by casting out the 9's :

$$\left.\begin{array}{lll} 7845 & \text{excess of} & 9=6 \\ 6780 & \text{`` \quad ``} & 9=3 \\ 8768 & \text{`` \quad ``} & 9=2 \\ 5343 & \text{`` \quad ``} & 9=6 \\ 3987 & \text{`` \quad ``} & 9=0 \end{array}\right\}\text{Excess of 9's}=8.$$

$$\overline{32723 \text{ excess of } 9=8}$$

Explanation.—Adding the digits in the first number, we have 24. Dividing by 9, we have 6 for a remainder, which is the excess of the 9's. Treating the remaining numbers in the same manner, we obtain the excesses 3, 2, 6, 0. Adding the excesses and taking the excess of their sum, we have 8; this being equal to the excess of the sum the work is correct.

3. Add the following, beginning at the left:

$$\begin{array}{r} 8456 \\ 9799 \\ 4363 \\ 5809 \\ 5432 \\ \hline 31 \\ 26 \\ 23 \\ 29 \\ \hline 33859 \end{array}$$

From this operation, we see that it is more convenient to begin at the right

Remark.—We can not add 8 apples and 5 peaches because we can not express the result in either denomination. Only numbers of the same name can be added.

EXAMPLES.

1. Add the numbers comprised between 20980189 and 20980197.

2. 6095054+900703+90300420+9890655+37699+29753= what?

3. Add the following, beginning at the left: 97674; 347-893; 789356; 98935679; 123456789.

4. Add all the prime numbers between 1 and 107 inclusive.

5. Add 31989, 63060, 132991, 1280340, 987654321, 78903, and prove the result by casting out the 9's.

6. Add the consecutive numbers from 100 to 130.

7. Add the numbers from 9897 to 9910 inclusive.

8. Add MDCCCLXXVI, MDCXCVIII, DCCCCXLIX, DCCCLXII.

CHAPTER IV.

SUBTRACTION.

1. ***Subtraction*** is the process of finding the difference between two numbers.

2. Subtract the following and prove the result by casting out the 9's:

$$\begin{array}{rl} 984895 & \text{excess of 9's}=7 \\ 795943 & \text{“ \quad “ 9's}=1 \\ \hline 188952 & \text{“ \quad “ 9's}=6 \end{array}\Bigg\}\text{Excess of 9's}=7.$$

Explanation.—Adding the digits in the first number, we have 43. Dividing by 9 the remainder is 7, which is the excess of the 9's. Treating the subtrahend and remainder in the same manner, we have the excesses 1 and 6. But subtraction is the opposite of addition and since the minuend is equal to the sum of the subtrahend and remainder, the excess of the sum of the excesses in the subtrahend and remainder is equal to the excess in the minuend. This is the same proof as that required if we were to add the subtrahend and remainder.

3. We begin at the right to subtract, so that if a figure of the subtrahend is greater than that corresponding to it in the minuend, we can borrow one from the next higher denomination and reduce it to the required denomination and then subtract.

4. Subtract the following and illustrate the process:

$$\begin{array}{r} {\scriptstyle 1=9\ 9\ 9\ 9\ 9\ 9\ 9+1} \\ 90000000 \\ 85784895 \\ \hline 4215105 \end{array} \qquad \begin{array}{r} {\scriptstyle 1=9\ 9\ 9\ 9\ 9\ 9+1} \\ 9856342 \\ 8978567 \\ \hline 877775 \end{array} \Big\} \text{Add.} \qquad \begin{array}{r} {\scriptstyle 1=9\ 9\ 9+1} \\ 4326546 \\ 3214957 \\ \hline 1111589 \end{array} \Big\} \text{Add.}$$

EXAMPLES.

1. From 9347893987 take 8968935789. Prove the result by casting out the 9's.

2. 7847893578—6759984699=what?

Which is the nearer number to 920864; 1816090 or 27497?

4. 34567—34518+3—2+3—4+7+18—567+43812—1326+678=what.?

5. 5+6+7—12—13+14—2—3+7—8—6+5+12—8=what?

6. 3+4—(6+7)—8+27—(1+3—2—3) — (7—8+5) 3+7=what?

CHAPTER V.

MULTIPLICATION.

1. ***Multiplication*** is the process of taking one number as many times as there are units in another; or it is a short method of addition when the numbers to be added are equal.

2. Multiply the following and prove the result by casting out the 9's:

$$\begin{array}{rl} 7855 & \text{excess of 9's}=7 \\ 435 & \text{" \quad " 9's}=3 \\ \hline 39275 & \qquad\qquad \overline{21} \text{ excess of 9's}=3. \\ 23565 & \\ 31420 & \\ \hline 3416925 & =\text{excess of 9's}=3. \end{array}$$

Explanation.—Adding the digits in the multiplicand and dividing the sum by 9, the remainder is 7 which is the excess of the 9's. Adding the digits in the multiplier and dividing the sum by 9, we have the remainder 3 which is the excess of the 9's. Now, since multiplication is a short method of addition when the numbers to be added are equal, we multiply the excess in the multiplicand by the excess in the multiplier and find the excess, and this being equal to the excess in the product, the work is correct.

3. Multiply the following, beginning at the left:

```
         75645
         765
        ------
1st.... 49
         35
          42
2d..... 4228
          3035
           36
3d...... 3524
           2530
             30
              20
               25
        --------
        57868425
```

3. From this operation, we see that it is more convenient to begin at the right to multiply.

5. In multiplication, the multiplicand may be abstract, or concrete; but the multiplier is always abstract.

6. The sign of multiplication is $\times$, and is read, *multiplied by*, or *times*. When this sign is placed between two numbers it denotes that one is to be multiplied by the other. In this case, it has not been established which shall be the multiplicand and which the multiplier. Thus $8\times5=40$, either may be considered the multiplicand and the other the multiplier. If 8 is the multiplicand, we say, 8 multiplied by 5 equals 40, but if 5 is the multiplicand we say, 8 *times* 5 equals 40.

EXAMPLES.

1. $562402\times345728=$what?

2. 1 mile $=$ 63360 inches; how many inches from the earth to the moon the distance being 239000 miles?

3. Multiply 789627 by 834, beginning at the left to multiply.

4. 1 acre $=$ 43560 sq. in.; how many square inches in a field containing 427 acres?

5. Multiply 6934789643 by 34789. Prove the result by casting out the 9's.

6. $2778588\times34678=$what?

7. $2\times3\times4-3\times7+3-2\times2+4+8\times2+4-3\times5+27=$what?

8. $5\times7+6\times7+8\times7-4\times6+6\times6+7\times6=$what?

9. $356789\times4876=$what?

10. $395076\times576426=$what?

11. $7733447\times998800=$what?

12. $5654321\times999880=$what?

CHAPTER VI.

DIVISION.

1. ***Division*** is the process of finding how many times one number is contained in another; or, it is a short method of subtraction when the numbers to be subtracted are equal.

2. Divide the following and prove the result by casting out the 9's:

$$
\begin{array}{r}
67)5484888(81864 \\
\underline{536} \\
124 \\
\underline{67} \\
578 \\
\underline{536} \\
428 \\
\underline{402} \\
268 \\
\underline{268}
\end{array}
$$

Dividend
5484888 excess of 9's=0.

$$
\left.\begin{array}{r}
\text{Quotient} \\
\text{81864 excess of 9's=0} \\
\text{Divisor} \\
\text{67 excess of 9's=4}
\end{array}\right\}
\begin{array}{c}
\text{Excess of 9's} \\
\text{in this product} \\
\text{equals 0.}
\end{array}
$$

Explanation.—Adding the digits in the dividend and dividing the sum by 9, we have the remainder 0, which is the excess of the 9's. Adding the digits in the quotient and dividing the sum by 9, we have the remainder 0, which is the excess of the 9's in the quotient. Adding the digits in the divisor and dividing the sum by 9, we have the remainder 4, which is the excess of the 9's in the divisor. Since division is the reverse of multiplication, the quotient corresponding to the multiplicand, the divisor to the multiplier, and the dividend to the product, we multiply the excess in the quotient by the excess in the divisor. The excess of this product is 0. This excess being equal to the excess of the 9's in the dividend, the work is correct.

If there be a remainder after dividing, find its excess and add it to the excess of the product of the excesses of the quotient and divisor. Take the excess of the sum and if it is equal to the excess of the dividend the work is correct.

3. The sign of division is $\div$, and is read *divided by.*

4. When the divisor and dividend are of the same denomination the quotient is abstract; but when of different denominations, the divisor is abstract and the quotient is the same as the dividend. Thus, 24 ct. $\div$ 4ct. $=6$, and 24 ct. $\div 4 = 6$ ct.

Remark.—We begin at the left to divide, that after finding how many times the divisor is contained in the fewest left-hand figures of the dividend, if there be a remainder we can reduce it to the next lower denomination and find how many times the divisor is contained in it, and so on.

Note.—The proof by casting out the 9's will not rectify errors caused by inserting or omitting a 9 or a 0, or by interchanging digits.

EXAMPLES.

1. $4326422 \div 961 =$ what? Prove the result by casting out the 9's.

2. $245379633477 \div 1263 =$ what? Prove the result by casting out the 9's.

3. What number multiplied by 109 with 98 added to the product, will give 106700?

4. The product of two numbers is 212492745; one of the numbers is 1035; what is the other number?

5. $27 \div 9 \times 3 \div 9 - 1 + 3 \div 3 \times 9 - 8 \div 4 + 5 \times 2 - 3 \times 2 \div 2 \div 3 - (3 \times 4 \div 6 + 5 - 2) + 81 \div 27 \times 3 \div 9 \times 18 \div 6 =$ what? [Hint.— Perform the operations indicated by the multiplication and division signs in the exact order of their occurrence.]

6. $(64 \div 32 \times 96 \div 12 - 7 - 5 + 3) \times \{[(27 \div 3) \div 9 - 1 + 2] + 91 \div 13 \times 7 - 45\} \times 9 + 45 \div 9 + 3 - 1 =$ what?

7. $2 \times 2 \div 2 \div 2 \div 2 \times 2 \times 2 \div 2 \div 2 \div 2 =$ what? *Ans.* $\frac{1}{4}$.

8. $3 \div 3 \div 3 \times 3 \times 3 \div 3 \div 0 \times 4 \times 4 \times 5 \times 5 =$ what? *Ans.* ∞.

9. $2 \times 2 \times 2 \div 2 \times 2 \div 2 \div 2 \times 2 \times 2 \times 0 \times 2 \times 2 =$ what? *Ans.* 0.

CHAPTER VII.

COMPOUND NUMBERS.

1. ***A Compound Number*** is a number which expresses several different units of the same kind of quantity.

2. ***A Denominate Number*** is a concrete number in which the unit is a measure; as, 5 *feet*, 7 *pints*.

3. ***The Terms*** of a compound number are the numbers of its different units. Thus, in 4 bu. 3 pk. 7 qt. 1 pt., the terms are 4 bu. and 3 pk. and 7 qt. and 1 pt.

4. ***Reduction of Compound Numbers*** is the process of changing a compound number from one denomination to another. There are Two Cases, *Reduction Descending* and *Reduction Ascending*.

5. ***Reduction Descending*** is the process of reducing a number from a higher to a lower denomination.

6. ***Reduction Ascending*** is the process of reducing a number from a lower to a higher denomination.

Ex. Reduce 2 E. Fr. 1 E. En. 2 E. Fl. 3 yd. 2 na. to nails.

					6	
					5	
					4+$1\frac{1}{5}$ in.	
					3	
					4	4
E. Fr.	E. En.	E. Sc.	E. Fl.	yd.	qr.	na.
2	1		2	3		2
6	5		3	4		
12qr.	5qr.		6qr.	12 qr.		

12 qr.
6 "
5 "
12 "
35 qr.
4
140 na.
2 "
142 na.

TIME MEASURE.

1. ***Time*** is a measured portion of duration.

2. The *measures* of time are fixed by the rotation of the earth on its axis and its revolution around the sun.

3. ***A Day*** is the time of one rotation of the earth on its axis.

4. ***A Year*** is the time of one revolution of the earth around the sun.

TABLE.

60 seconds (sec.)	make	1 minute (min.)
60 minutes	"	1 hour (hr.)
24 hours	"	1 day (da.)
7 days	"	1 week (wk.)
4 weeks	"	1 lunar month (mo.)
13 lunar months, 1 da. 6 hr.	"	1 year (yr.)
12 calender months	"	1 year.
365 days	"	1 common year.
365 da. 5 hr. 48 min. 46.05 sec.	"	1 solar year.
365 da. 6 hr. 9 min. 9 sec.	"	1 sidereal year.
365 da. 6 hr. 13 min. 45.6 sec.	"	1 Anomalistic year.
366 days	"	1 leap year, or bissextile year.
354 days	"	1 lunar year.
19 years	"	1 Metonic cycle.
28 years	"	1 solar cycle.
15 years	"	1 Cycle of Indiction.
532 years	"	1 Dionysian Period.

5. The *unit of time* is the day.

6. ***The Sidereal Day*** is the exact time of one rotation of the earth on its axis. It equals 23 hr. 56 min. 4.09 sec.

7. ***The Solar Day*** is the time between two successive appearances of the sun on a given meridian.

8. ***The Astronomical Day*** is the solar day, beginning and ending at noon.

9. ***The Civil Day,*** or ***Mean Solar Day,*** is the average of all the solar days of the year. It equals 24 hr. 3 min. 56.556 sec.

10. ***The Solar Year,*** or ***Tropical Year,*** is the time between two successive passages of the sun through the vernal equinox.

11. ***The Sidereal Year*** is the time of a complete revolution of the earth about the sun, measured by a fixed star.

12. ***The Anomalistic Year*** is the time of two successive passages of the earth through its perihelion.

13. ***A Lunar Year*** is 12 lunar months and consists of 354 day.

14. ***A Metonic Cycle*** is a period of 19 solar years, after which the new moons again happen on the same days of the year.

15. ***A Solar Cycle*** is a period of 28 solar years, after which the first day of the year is restored to the same day of the week. To find the year of the cycle, we have the following rule:

Add nine to the date, divide the sum by twenty-eight; the quotient is the number of cycles, and the remainder is the year of the cycle. Should there be no remainder the proposed year is the twenty-eighth, or last of the cycle. The formula for the above rule is $\left\{\frac{x+9}{28}\right\}_r$ in which x denotes the date, and r the remainder which arises by dividing $\chi+9$ by 28, is the number required.

Thus, for 1892, we have $(1892+9)\div28=67\frac{25}{28}$. $\therefore$ 1892 is the 25th year of the 68 cycle.

16. ***The Lunar Cycle*** is a period of 19 years, after which the new moons are restored to the same day of the civil month.

The new moon will fall on the same days in any two years which occupy the same place in the cycle; hence, a table of the moon's phases for 19 years will serve for any year whatever when we know its number in the cycle. This number is called the *Golden Number.*

To find the Golden Number: *Add 1 to the date, divide the sum by 19; the quotient is the number of the cycle elapsed and the remainder is the Golden Number.*

The formula for the same is $\left\{\frac{x+1}{19}\right\}_r$ in which r is the remainder after dividing the date$+1$ by 19. It is the Golden Number.

17. ***A Dionysian*** or ***Paschal Period*** is a period of 532 year, after which the new moons again occur on the same day of the month and the same day of the week. It is obtained by multiplying a *Lunar Cycle* by a *Solar Cycle.*

18. ***A Cycle of Indiction*** is a period of 15 years, at the end of which certain judicial acts took place under the Greek emperors.

19. ***Epact*** is a word employed in the calender to signify the moon's age at the beginning of the year.

The common solar year, containing 365 days, and the lunar year only 354, the difference is 11 days; whence, if a new moon fall on the first of January in any year, the moon will be 11 days old on the first day of the following year, and 22 days on the first of the third year. The *epact* of these years are, *therefore,* eleven and twenty-two respectively. Another addition of eleven

days would give thirty-three for the *epact* of the fourth year; but in consequence of the insertion of the intercalary month in each third year of the lunar cycle, this epact is reduced to three. In like manner the epacts of all the following years of the cycle are obtained by successively adding eleven to the epact of the former year, and rejecting thirty as often as the sum exceeds that number.

SOLUTIONS.

Ex. 2.—Reduce 2 p. 3 pn. 1 tr. 1 hhd. 1 gal. 1 qt. to pints.

						126		
						84		
						42		
						$31\frac{1}{2}$		
					4	63	4	2
p.	pn.	tr.	bbl.	T.	hhd.	gal.	qt.	pt.
2	3	1			1	1	1	
126	84	42			63	63		
252 gal.	252 gal.	42 gal.			63 gal.			
						42		
						252		
						252		
						610 gal.		
						4		
						2440 qt.		
						1 "		
						2441 "		
						2		
						4882 pt.		

Ex. 3. I. Reduce 2 bu. 3 pk. 2 qt. 1 pt. to pints.

Equation Method.

Solution: II. $\begin{cases} 1. & 1 \text{ bu.}=4 \text{ pk.} \\ 2. & 2 \text{ bu.}=2\times4 \text{ pk.}=8 \text{ pk.} \\ 3. & 8 \text{ pk.}+3 \text{ pk.}=11 \text{ pk.} \\ 4. & 1 \text{ pk.}=8 \text{ qt.} \\ 5. & 11 \text{ pk.}=11\times8 \text{qt.}=88 \text{ qt.} \\ 6. & 88 \text{ qt.}+2 \text{ qt.}=90 \text{ qt.} \\ 7. & 1 \text{qt.}=2 \text{ pt.} \\ 8. & 90 \text{ qt.}=90\times2 \text{ pt.}=180 \text{ pt.} \\ 9. & 180 \text{ pt.}+1 \text{ pt.}=181 \text{ pints.} \end{cases}$

Conclusion: III. $\therefore$ 2 bu. 3 pk. 2 qt. 1 pt.=181 pints.

Ex. 4. I. Reduce 529 pints to bushels.

Equation method.

Solution: II. $\begin{cases} 1. & 2 \text{ pt.}=1 \text{ qt.} \\ 2. & 529 \text{ pt.}=529\div2=264 \text{ qt.}+1 \text{ pt.} \\ 3. & 8 \text{ qt.}=1 \text{ pk.} \\ 4. & 264 \text{ qt.}=264\div8=33 \text{ pk.} \\ 5. & 4 \text{ pk.}=1 \text{ bu.} \\ 6. & 33 \text{ pk.}=33\div4=8 \text{ bu.}+1 \text{ pk.} \end{cases}$

Conclusion: III. $\therefore$ 529 pints=8 bu. 1 pk. 1 pt.

Ex. 5. How many gallons will a tank 4 ft. long, 3 ft. wide, and 1 ft. 8 in. deep contain?

Solution : II. $\begin{cases} 1. & 4 \text{ ft.}=\text{length}, \\ 2. & 3 \text{ ft.}=\text{width, and} \\ 3. & 1 \text{ ft. } 8 \text{ in.}=1\frac{2}{3} \text{ ft.}=\text{depth.} \\ 4. & 4\times3\times1\frac{2}{3}=20 \text{ cubic ft.}=\text{contents of tank.} \\ 5. & 1 \text{ cu. ft.}=1728 \text{ cu. in.} \\ 6. & 20 \text{ cu. ft.}=20\times1728 \text{ cu. in.}=34560 \text{ cu. in.} \\ 7. & 231 \text{ cu. in.}=1 \text{ gal.} \\ 8. & \therefore 34560 \text{ cu. in.}=34560\div231=149\frac{47}{77} \text{ gal.} \end{cases}$

Conclusion : III. $\therefore$ The tank will contain $149\frac{47}{77}$ gallons.

(*Fish's Comp. Arith., p. 126, prob. 2.*)

EXAMPLES.

1. How many links in 46 mi. 3 fur. 5 ch. 25 links?

2. How many acres in a field containing 1377 square chains?

3. How many cubic inches in 29 cords of wood?

4. In 1436 nails how many Ell English?

5. How many miles in 3136320 inches?

6. In 47 ℔. 2 ℥ 3 ʒ 1 ℈ 19 gr. how many grains?

7. Change 16 lb. 3 oz. 1 gr., Troy weight to Avoirdupois weight.

8. An apothecary bought by Avoirdupois weight, 2 ℔. 8 oz. of quinine at $2.40 per ounce, which he retailed at 20 ct. a scruple. What was his gain on the whole?

9. How many seconds in a Dionysian Period?

10. How many seconds in the month of February, 1892.

11. How many seconds in the circumference of a wagon wheel?

12. How long would it take a body to move from the earth to the moon, moving at the rate of 30 miles per day.

13. If a man travels 4 miles per hour, how far can he travel in 2 weeks and 3 days?

14. How much may be gained by buying 2 hogsheads of molasses, at 40 ct. per gallon, and selling it at 12 cents per quart?
Ans. \$10.08

15. In 74726807872 seconds, how many solar years?
Ans. 2368 years.

16. At \$4 per quintal, how many pounds of fish may be bought for \$50.24? *Ans.* 1256 pounds.

17. How many bottles of 3 pints each will it take to fill a hogshead? *Ans.* 168.

18. What will 73 bushels of meal cost, at 2 cents per quart?
Ans. \$46.72.

19. How many ounces of gold are equal in weight to 6 ℔. of lead? *Ans.* $87\frac{1}{2}$ oz.

20. What is the difference between the weight of $42\frac{3}{8}$ ℔. of iron and 42.375 ℔. of gold? *Ans.* 52545 gr.

21. How many bushels of corn will a vat hold that holds 5000 gallons of water. *Ans.* $537\frac{7}{64}$ bu.

22. A cellar 40 ft. long, 20 ft. wide and 8 ft. deep is half full of water. What will it cost to pump it out, at 6 cents a hogshead? *Ans.* \$22.797+.

23. If a man buys 10 bu. of chestnuts at \$5 a bushel, dry measure, and sells the same at 25 cents a quart, liquid measure, how much does he gain? *Ans.* \$43.09+ gain.

24. How many steps, 2 ft. 8 in. each, will a man take in walking a distance of 15 miles? *Ans.* 29700.

25. How many hair's width in a 40 ft. pole, if 48 hair's width equals 1 line?

26. How many chests of tea, weighing 24 pounds each, at 43 cents a pound, can be bought for \$1548? *Ans.* 150 chests.

27. How long will it take to count 6 million, at the rate of 80 a minute, counting 10 hours a day? *Ans.* 125 days.

28. How long will it take to count a billion, at the rate of 80 a minute, counting 12 hours a day? *Ans.*——

29. What will 15 hogsheads of beer cost, at 3 cents a pint.
Ans. \$194.40.

30. How many shingles will it take to cover the roof of a building 60 ft. long and 56 ft. wide, allowing each shingle to be 4 inches wide and 18 inches long, and to lie $\frac{1}{3}$ to the weather?
Ans. 20160.

31. There are 9 oz. of iron in the blood of 1 man. How many men would furnish iron enough in their veins to make a plowshare weighing $22\frac{1}{2}$ lbs.? *Ans.* 40.

CHAPTER VIII.

GREATEST COMMON DIVISOR.

1. ***A Divisor*** of a number is a number that will exactly divide it.

2. ***A Common Divisor*** of two or more numbers is a number that will exactly divide each of them.

3. ***The Greatest Common Divisor,*** or ***Highest Common Factor,*** of two or more numbers is the greatest number that will exactly divide each of them.

I. Find the G. C. D. of 60, 120, 150, 180.

$$\text{II.}\begin{cases}1.\ 60=2\times2\times3\times5.\\2.\ 120=2\times2\times2\times3\times5.\\3.\ 150=2\times3\times5\times5.\\4.\ 180=2\times2\times3\times3\times5.\\5.\ \text{G. C. D.}=2\times3\times5=30.\end{cases}$$

III. ∴ G. C. D.$=30$.

Explanation.—By inspecting the factors of each number we observe that 2 is found in each set of factors; hence, each of the numbers can be divided by 2. But only once, since it is found only once in the factors of 150. We also observe that 3 will divide the numbers only once, since it occurs only once in the factors of 60 and 120. Also, 5 will divide them but once, since 60, 120 and 180 contain it but once. Hence, the numbers, 60, 120, 150, 180, being divisible by 2, 3 and 5, are divisible by their product, $2\times3\times5=30$.

I. Find the G. C. D. of 180, 1260, 1980.

$$\text{II.}\begin{cases}1.\ 180=2\times2\times3\times3\times5.\\2.\ 1260=2\times2\times3\times3\times5\times7.\\3.\ 1980=2\times2\times3\times3\times5\times11.\\4.\ \text{G. C. D.}=2\times2\times3\times3\times5=180.\end{cases}$$

III. ∴ G. C. D. of 180, 1260, 1980$=180$.

Explanation.—2 being found twice in each number, they are each divisible by 2×2 or 4; also 3 being found twice in each number, they are each divisible by 3×3 or 9. 5 being found in each number, they are each divisible by 5. Hence, they are divisible by the product of these factors, $2\times2\times3\times3\times5=180$.

EXAMPLES.

1. Find the G. C. D. of 78, 234, and 468.

2. What is the G. C. D. of 36, 66, 198, 264, 600 and 720?

3. I have three fields: the first containing 16 acres, the second 20 acres, and the third 24 acres. What is the largest sized lots

containing each an exact number of acres, into which the whole can be divided? *Ans.* 4 A. lots.

4. A farmer has 12 bu. of oats, 18 bu. of rye, 24 bu. of corn and 30 bu. of wheat. What are the largest bins of uniform size, and containing an exact number of bushels, into which the whole can be put, each kind by itself, and all the bins be full? *Ans.* 6 bu. bins.

5. A has a four-sided field whose sides are 256, 292, 384, and 400 feet respectively; what is the length of the rails used to fence it, if they are all of equal length and the longest that can be used? *Ans.* 4 ft.

6. In a triangular field whose sides are 288, 450, and 390 feet respectively, how many rails will it require to fence it, if the fence is 5 rails high, and what must be the length of the rails if they lap over one foot? *Ans.* Length of rail, 7 ft. No. 940.

CHAPTER IX.

LEAST COMMON MULTIPLE.

1. ***A Multiple*** of a number is a number that will exactly contain it; thus, 24 is a multiple of 6.

2. ***A Common Multiple*** of two or more numbers is a number that will exactly contain each of them.

3. ***The Least Common Multiple*** of two or more numbers is the least number that will exactly contain each of them.

I. Find the L. C. M. of 30, 40, 50.

II.
1. $30=2\times3\times5$.
2. $40=2\times2\times2\times5$.
3. $50=2\times5\times5$.
4. L. C. M.$=2\times2\times2\times3\times5\times5=600$.

III. ∴ L. C. M. of 30, 40, 50=600.

Explanation.—The L. C. M. must contain 2 three times, or it would not contain 40; it must contain 5 twice, or it would not contain 50; it must contain 3 once, or it would not contain 30. Since all the factors of the numbers, 30, 40, 50, are contained in the L. C. M., it will contain each of them without a remainder.

I. Find the L. C. M. of 2310, 210, 30, 6.

II.
1. $2310=2\times3\times5\times7\times11$.
2. $210=2\times3\times5\times7$.
3. $30=2\times3\times5$.
4. $6=2\times3$.
5. L. C. M.$=2\times3\times5\times7\times11=2310$.

III. ∴ L. C. M. of 2310, 210, 30, 6=2310.

Explanation.—2 and 3 must be used, else the L. C. M. would not contain 6. 2, 3, and 5 must be used, else the L. C. M. would not contain 30. Hence 5 must be taken with the factors of 6. In like manner 7 must be taken with the factors already taken, else the L. C. M. would not contain 210. The factor 11 must be taken with those already taken, else the L. C. M. would not contain 2310. Hence 2, 3, 5, 7, and 11 are the factors to be taken and their product 2310 is the L. C. M.

I. The product of the L. C. M. of three numbers between 1 and 100 is 6804; and the quotient of the L. C. M. divided by the G. C. D. is 84. What are the numbers?

II.
1. L. C. M. $\times$ G. C. D. $=6804$, and
2. $\frac{\text{L. C. M.}}{\text{G. C. D.}}=84$.
3. $\therefore$ L. C. M. $\times$ G. C. D. $\div\frac{\text{L. C. M.}}{\text{G. C. D.}}=$ L. C. M. $\times$ G. C. D. $\times\frac{\text{G. C. D.}}{\text{L. C. M.}}=(\text{G. C. D.})^2=6804\div84=81$.
4. G. C. D. $=\sqrt{81}=9$, by extracting the square root.
5. $\therefore$ L. C. M. $=6804\div9=756$.
6. $9=3\times3$.
7. $756=2\times2\times3\times3\times3\times7$.
8. $3\times3\times2\times2=36$.
9. $3\times3\times3\times2=54$.
10. $3\times3\times7\quad=63$.

III. $\therefore$ 36, 54, and 63 = the numbers.

Explanation.—Since 9 is the G. C. D., each of the numbers contains the factors of 9. Since there are two 2's in the L. C. M., one of the numbers must contain these factors. In like manner one of the numbers must contain three 3's; one of them must also contain 7. $\therefore$ We write two 3's for each of the numbers, two 2's to any set of these 3's, and 3 and 7 with either of the remaining sets, observing that the product of the factors in any set does not exceed 100. If we omit 2 in step 9, the product of the factors is 27. Hence 27, 36, 63 are numbers also satisfying the conditions of the problem.

EXAMPLES.

1. What is the L. C. M. of 13, 14, 28, 39, and 42?

2. What is the L. C. M. of 6, 8, 10, 18, 20, 36, and 48?

3. What is the L. C. M. of 18, 24, 36, 126, 20, 48, 96, 720, and 84?

4. What is the smallest sum of money with which I can purchase a number of oxen at \$50 each, cows at \$40 each, or horses at \$75 each? *Ans.* \$600.

5. Find three numbers whose L. C. M. is 840 and G. C. D. 42. *Ans.* 84, 210, and 420.

6. What three numbers between 30 and 140 having 12 for their G. C. D. and 2772 for their L. C. M.? *Ans.* 36, 84, and 132.

7. At noon the second, minute, and hour hands of a clock are together; how long after will they be together again for the first time?

8. J. S. H. has 5 pieces of land; the first containing 3 A. 2 rd. 1 p.; the second, 5 A. 3 rd. 15 p.; the third 8 A. 29 p.; the fourth, 12 A. 3 rd. 17 p.; and the fifth, 15 A. 31 p. Required the largest sized house-lots, containing each an exact number of square rods, into which the whole may be divided.

Ans. 1 A. 21 p.

9. The product of the L. C. M. of three numbers by their G. C. D.=864, and the L. C. M. divided by the G. C. D.=24; find the numbers. *Ans.* 12, 18, and 48.

CHAPTER X.

FRACTIONS.

1. ***A Fraction*** is a number of the equal parts of a unit.

2. Fraction.
- 1. Common, or Vulgar.
  - 1. As to Form.
    - 1. Simple.
    - 2. Complex.
    - 3. Compound.
  - 2. As to Value.
    - 1. Proper.
    - 2. Improper.
    - 3. Mixed.
- 2. Decimal.
  - 1. Pure.
  - 2. Mixed.
  - 3. Circulating.
    - 1. Pure.
    - 2. Mixed.
- 3. Continued Fractions.

3. ***A Common Fraction,*** or ***Vulgar Fraction,*** is one in which the unit is divided into *any number* of equal parts; and is expressed by two numbers, one written above the other, with a horizontal line between them. Thus, $\frac{5}{6}$ expresses five-sixths.

4. ***A Simple Fraction*** is a fraction having a single integral numerator and denominator; as, $\frac{3}{5}$.

5. ***A Complex Fraction*** is a fraction whose numerator, or denominator, or both, are fractional; as, $\frac{4}{3\frac{1}{2}}$, $\frac{2\frac{1}{2}}{3\frac{1}{3}}$, $\frac{2\frac{1}{3}}{5}$.

6. ***A Compound Fraction*** is a fraction of a fraction; as, $\frac{2}{3}$ of $\frac{3}{4}$.

7. ***A Proper Fraction*** is a simple fraction whose numerator is less than its denominator; as, $\frac{4}{5}$.

8. ***An Improper Fraction*** is a simple fraction whose numerator is greater than its denominator; as, $\frac{5}{4}$.

9. ***A Mixed Number*** is a whole number and a fraction; as, $3\frac{3}{4}$.

10. ***A Decimal Fraction*** is a fraction whose denominator is ten, or some power of ten; as, $\frac{3}{10}$, $\frac{4}{100}$, $\frac{27}{1000}$. The denominator of a decimal is usually omitted and the point (.) is used to determine the value of the decimal expression. Thus, $\frac{3}{10}=.3$, $\frac{27}{1000}=.027$.

11. ***A Pure Decimal*** is one which consists of decimal figures only; as, .375.

12. ***A Mixed Decimal*** is one which consists of an integer and a decimal; as, 5.25.

13. ***A Circulating Decimal***, or a ***Circulate***, is a decimal in which one or more figures are repeated in the same order; as, .2121 etc. When a common fraction is in its lowest terms and the denominator contains factors other than 2 or powers of 2, and 5 or powers of 5, the equivalent decimal fraction will be circulating. Thus, $\frac{7}{1500}=\frac{7}{2^2\times3\times5^3}$ will, when reduced to a decimal, be circulating because the denominator contains the factor 3.

The repeating figure or set of figures is called a *Repetend*, and is indicated by placing a dot over the first and the last figure repeated.

14. ***A Pure Circulate*** is one which contains no figures but those which are repeated; as, $.\dot{2}7\dot{3}$.

15. ***A Mixed Circulate*** is one which contains one or more figures before the repeating part; as, $.45\dot{3}4\dot{2}$.

16. ***A Simple Repetend*** contains but one figure; as, $.\dot{3}$.

17. ***A Compound Repetend*** contains more than one figure; as, $\dot{3}5\dot{4}$.

18. ***Similar Repetends*** are those which begin and end at the same decimal places; as, $.3\dot{4}6\dot{7}$, and $.0\dot{3}5\dot{8}$.

19. ***Dissimilar Repetends*** are those which begin or end at different decimal places; as, $.5\dot{3}\dot{6}$, $.8\dot{3}\dot{5}$, and $.3\dot{5}6\dot{7}$.

20. ***A Perfect Repetend*** is one which contains as many decimal places, less 1, as there are units in the denominator of the equivalent common fraction; thus, $\frac{1}{7}=.\dot{1}4285\dot{7}$.

21. ***Conterminous Repetends*** are those which end at the same decimal place; as, $.42\dot{6}\dot{7}$, $.\dot{3}27\dot{5}$, and $.03\dot{2}\dot{1}$.

22. ***Co-originous Repetends*** are those which begin at the same decimal place; as, $.3\dot{7}\dot{8}$, $.5\dot{6}2\dot{4}$, and $3.6\dot{2}\dot{3}$.

I. Reduce $\frac{9}{12}$ to its lowest terms.

$3)\frac{9}{12}=\frac{3}{4}$.

Explanation.—Dividing the numerator 9, by 3, without changing the denominator, the value of the fraction is diminished as many times as there are units in the divisor 3. Dividing the denominator 12, by 3, without changing the numerator 9, the value of the fraction is increased as many times as there are units in the divisor 3. Hence, if we divide both terms by 3, the increase by dividing the denominator will be equal to the decrease by dividing the numerator, and the value of the fraction will remain unchanged.

I. Reduce $\frac{2}{3}$ to a higher denomination.

$\frac{2}{3}\times\frac{4}{4}=\frac{8}{12}$.

Explanation.—Multiplying the numerator 2, by 4, without changing the denominator, the value of the fraction is increased as many times as there are units in the multiplier 4. Multiplying the denominator 3, by 4, without changing the numerator, the value of the fraction is decreased as many times as there are units in the multiplier 4. Hence, if we multiply both terms by 4, the increase by multiplying the numerator is equal to the decrease by multiplying the denominator, and the value of the fraction remains unchanged.

I. Reduce $9\frac{7}{8}$ to an improper fraction.

$$\text{Solution: II.}\begin{cases} 1. & 9\frac{7}{8}=9+\frac{7}{8}. \\ 2. & 1=\frac{8}{8}. \\ 3. & 9=9\times\frac{8}{8}=\frac{72}{8}. \\ 4. & \frac{72}{8}+\frac{7}{8}=\frac{79}{8}. \end{cases}$$

Conclusion: III. $\therefore 9\frac{7}{8}=\frac{79}{8}$.

I. Reduce $\frac{5}{8}$ to 24ths.

$$\text{II.}\begin{cases} 1. & \frac{8}{8}=\frac{24}{24}. \\ 2. & \frac{1}{8}=\frac{1}{8}\text{ of }\frac{\overset{3}{24}}{24}=\frac{3}{24}. \\ 3. & \frac{5}{8}=5\text{ times }\frac{3}{24}=\frac{15}{24}. \end{cases}$$

III. $\therefore \frac{5}{8}=\frac{15}{24}$.

I. Reduce $\frac{5}{6}$ to 8ths.

$$\text{II.}\begin{cases} 1. & \frac{6}{6}=\frac{8}{8} \\ 2. & \frac{1}{6}=\frac{1}{6} \text{ of } \frac{8}{8}=\dfrac{\frac{8}{6}}{8}=\dfrac{\frac{4}{3}}{8}=\dfrac{1\frac{1}{3}}{8}. \\ 3. & \frac{5}{6}=5 \text{ times } \dfrac{1\frac{1}{3}}{8}=\dfrac{6\frac{2}{3}}{8}. \end{cases}$$

III. $\therefore \frac{5}{6}=\dfrac{6\frac{2}{3}}{8}.$

I. Reduce $\frac{1}{2}$ to 3ds.

$$\text{II.}\begin{cases} 1. & \frac{2}{2}=\frac{3}{3}. \\ 2. & \frac{1}{2}=\frac{1}{2} \text{ of } \frac{3}{3}=\dfrac{\frac{3}{2}}{3}=\dfrac{1\frac{1}{2}}{3}=1\frac{1}{2} \text{ thirds.} \end{cases}$$

III. $\therefore \frac{1}{2}=\dfrac{1\frac{1}{2}}{3}.$

Explanation.—In taking $\frac{1}{2}$ of $\frac{3}{3}$, we must divide 2 into the numerator. The denominator must be left unchanged; for that is the denomination to which the given fraction is to be reduced.

I. Reduce $\frac{3}{5}$ to 11ths.

$$\text{II.}\begin{cases} 1. & \frac{5}{5}=\frac{11}{11}. \\ 2. & \frac{1}{5}=\frac{1}{5} \text{ of } \frac{11}{11}=\dfrac{\frac{11}{5}}{11}=\dfrac{2\frac{1}{5}}{11}. \\ 3. & \frac{3}{5}=3 \text{ times} \dfrac{2\frac{1}{5}}{11}=\dfrac{6\frac{3}{5}}{11}. \end{cases}$$

III. $\therefore \frac{3}{5}=\dfrac{6\frac{3}{5}}{11}=6\frac{3}{5}$ elevenths.

I. Reduce $\frac{29}{3}$ to a mixed number.

$$\text{II.}\begin{cases} 1. & \frac{3}{3}=1. \\ 2. & \frac{29}{3}=29\div3=9\frac{2}{3}. \end{cases}$$

III. $\therefore \frac{29}{3}=9\frac{2}{3}.$

I. Reduce $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$ to their L. C. Denominator.

$$\text{II.}\begin{cases} 1. & \text{L. C. D.}=12. \\ 2. & \frac{3}{3}=\frac{12}{12}. \\ 3. & \frac{1}{3}=\frac{1}{3} \text{ of } \frac{12}{12}=\frac{4}{12}. \\ 4. & \frac{2}{3}=2\times\frac{4}{12}=\frac{8}{12}. \\ 5. & \frac{4}{4}=\frac{12}{12}. \\ 6. & \frac{1}{4}=\frac{1}{4} \text{ of } \frac{12}{12}=\frac{3}{12}. \\ 7. & \frac{3}{4}=3\times\frac{3}{12}=\frac{9}{12}. \\ 8. & \frac{6}{6}=\frac{12}{12}. \\ 9. & \frac{1}{6}=\frac{1}{6} \text{ of } \frac{12}{12}=\frac{2}{12}. \\ 10. & \frac{5}{6}=5\times\frac{2}{11}=\frac{10}{12}. \end{cases}$$

III. $\therefore \frac{2}{3}, \frac{3}{4}, \frac{5}{6}=\frac{8}{12}, \frac{9}{12}, \frac{10}{12}.$

I. Reduce $\frac{1}{2}$, $\frac{4}{5}$, $\frac{5}{8}$ to their L. C. Denominator.

$$\text{II.}\begin{cases}1. & \text{L. C. D.}=40.\\ 2. & 1=\frac{40}{40}.\\ 3. & \frac{1}{2}=\frac{1}{2}\times\frac{40}{40}=\frac{20}{40}.\\ 4. & \frac{4}{5}=\frac{4}{5}\times\frac{40}{40}=\frac{32}{40}.\\ 5. & \frac{5}{8}=\frac{5}{8}\times\frac{40}{40}=\frac{25}{40}.\end{cases}$$

III. $\therefore \frac{1}{2}, \frac{4}{5}, \frac{5}{8}=\frac{20}{40}, \frac{32}{40}, \frac{25}{40}$.

I. Add $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$.

$$\text{II.}\begin{cases}1. & \text{L. C. D.}=24.\\ 2. & 1=\frac{24}{24}.\\ 3. & \frac{3}{4}=\frac{3}{4}\times\frac{24}{24}=\frac{18}{24}.\\ 4. & \frac{5}{6}=\frac{5}{6}\times\frac{24}{24}=\frac{20}{24}.\\ 5. & \frac{7}{8}=\frac{7}{8}\times\frac{24}{24}=\frac{21}{24}.\\ 6 & \therefore \frac{3}{4}+\frac{5}{6}+\frac{7}{8}=\frac{18}{24}+\frac{20}{24}+\frac{21}{24}=\frac{59}{24}=2\frac{11}{24}.\end{cases}$$

III. $\therefore \frac{3}{4}+\frac{5}{6}+\frac{7}{8}=2\frac{11}{24}$.

I. Reduce $\frac{3}{4}$ to a fraction whose numerator is 15.

$$\text{II.}\begin{cases}1. & 1=\frac{15}{15}.\\ 2. & \frac{3}{4}=\frac{\cancel{3}}{4}\times\frac{15}{\underset{5}{\cancel{15}}}=\frac{15}{20}.\end{cases}$$

III. $\therefore \frac{3}{4}=\frac{15}{20}$.

I. Reduce $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$ to equivalent fractions having least common numerators.

$$\text{II.}\begin{cases}1. & \text{L. C. N.}=12.\\ 2. & 1=\frac{12}{12}.\\ 3. & \frac{2}{3}=\frac{\cancel{2}}{3}\times\frac{12}{\underset{6}{\cancel{12}}}=\frac{12}{18}.\\ 4. & \frac{3}{4}=\frac{\cancel{3}}{4}\times\frac{12}{\underset{4}{\cancel{12}}}=\frac{12}{16}.\\ 5. & \frac{4}{5}=\frac{\cancel{4}}{5}\times\frac{12}{\underset{3}{\cancel{12}}}=\frac{12}{15}.\end{cases}$$

III. $\therefore \frac{2}{3}, \frac{3}{4}, \frac{4}{5}=\frac{12}{18}, \frac{12}{16}, \frac{12}{15}$.

I. Subtract $\frac{5}{8}$ from $\frac{9}{10}$.

$$\text{II.}\begin{cases}1. & \text{L. C. D.}=40.\\ 2. & 1=\frac{40}{40}.\\ 3. & \frac{5}{8}=\frac{5}{8}\times\frac{40}{40}=\frac{25}{40}.\\ 4. & \frac{9}{10}=\frac{9}{10}\times\frac{40}{40}=\frac{36}{40}.\\ 5. & \therefore \frac{9}{10}-\frac{5}{8}=\frac{36}{40}-\frac{25}{40}=\frac{11}{40}.\end{cases}$$

III. $\therefore \frac{9}{10}-\frac{5}{8}=\frac{11}{40}$.

I. Multiply $\frac{7}{8}$ by $\frac{5}{6}$.

$$\text{II.}\begin{cases}1. & \frac{7}{8}\times\frac{6}{6}=\frac{7}{8}.\\ 2. & \frac{7}{8}\times\frac{1}{6}=\frac{1}{6}\text{ of }\frac{7}{8}=\frac{7}{48}.\\ 3. & \frac{7}{8}\times\frac{5}{6}=5\text{ times }\frac{7}{48}=\frac{35}{48}.\end{cases}$$

III. $\therefore \frac{7}{8}\times\frac{5}{6}=\frac{35}{48}$.

I. Divide $\frac{5}{8}$ by $\frac{3}{7}$.

$$\text{II.}\begin{cases} 1. & \frac{5}{8}\div\frac{7}{7}=\frac{5}{8}. \\ 2. & \frac{5}{8}\div\frac{1}{7}=7 \text{ times } \frac{5}{8}=\frac{35}{8}. \\ 3. & \frac{5}{8}\div\frac{3}{7}=\frac{1}{3} \text{ of } \frac{35}{8}=\frac{35}{24}=1\frac{11}{24}. \end{cases}$$

III. $\therefore\ \frac{5}{8}\div\frac{3}{7}=1\frac{11}{24}$.

Analysis to the last example:

$$\text{II.}\begin{cases} 1. & \frac{1}{7} \text{ is contained in 1, or } \frac{8}{8}, 7 \text{ times.} \\ 2. & \frac{3}{7} \text{ is contained in 1, or } \frac{8}{8},\ \frac{1}{3} \text{ of 7 times}=\frac{7}{3} \text{ times.} \\ 3. & \frac{3}{7} \text{ is contained in } \frac{1}{8},\ \frac{1}{8} \text{ of } \frac{7}{3} \text{ times}=\frac{7}{24} \text{ times.} \\ 4. & \frac{3}{7} \text{ is contained in } \frac{5}{8},\ 5 \text{ times } \frac{7}{24} \text{ times, or } \frac{35}{24} \text{ times.} \end{cases}$$

Note.—By inverting the divisor, we find how many times it is contained in 1.

EXAMPLES.

1. One-fifth equals how many twelfths?
2. Reduce $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, and $\frac{6}{7}$ to fractions having a common denominator.
3. Reduce $\frac{5}{8}$ to a fraction whose numerator is 13. *Ans.* $\dfrac{13}{11\frac{7}{8}}$.
4. Reduce $\frac{7}{8}$ to a fraction whose denominator is 11. *Ans.* $\dfrac{9\frac{5}{8}}{11}$.
5. Reduce $\frac{5}{6}$, $\frac{4}{7}$, $\frac{3}{8}$, to fractions having common numerators.
6. Add $\frac{1}{4}$, $\frac{7}{8}$, $\frac{11}{12}$, $\frac{5}{9}$, and $\frac{7}{11}$.
7. $\frac{3}{5}$ of $8\frac{3}{4}-\frac{2}{3}$ of $5=$what?
8. Multiply $\frac{3}{7}$ by $8\frac{3}{4}$.
9. Multiply $\frac{3}{4}$ of $9\frac{1}{4}$ of $\frac{8}{9}$ by $\frac{3}{4}$ of 17.
10. $\dfrac{12\frac{7}{8}}{15\frac{3}{4}}\div\dfrac{5\frac{2}{3}}{3\frac{2}{5}}=$what?
11. $\dfrac{11\frac{7}{8}-6\frac{3}{5}}{10\frac{9}{11}-9\frac{11}{12}}\div\dfrac{7\frac{5}{6}-5\frac{3}{4}}{8\frac{9}{10}-9\frac{4}{5}}=$what?
12. $\frac{25}{57}\times\frac{42}{65}\times\frac{19}{20}\div\frac{7}{13}=$what? - - - - - - - - - - *Ans.* $\frac{1}{2}$.
13. $$\begin{array}{c} 1 \\ \hline 2 \\ \hline 1 \\ \hline 3 \\ \hline 1 \\ \hline 4 \\ \hline 1 \\ \hline 5 \\ \hline 1 \\ \hline 6 \\ \hline 1 \\ \hline 7 \\ \hline 1 \\ \hline 8 \\ \hline 1 \\ \hline 9 \\ \hline 1 \\ \hline 10 \end{array}=\text{what?}$$ *Ans.* 907200.
14. $$\begin{array}{c} 1\frac{1}{2} \\ \hline 2\frac{1}{3} \\ \hline 3\frac{1}{4} \\ \hline 4\frac{1}{5} \\ \hline 5\frac{1}{6} \\ \hline 6\frac{1}{7} \\ \hline 2\frac{1}{2} \\ \hline 3\frac{1}{3} \\ \hline 4\frac{1}{4} \\ \hline 5\frac{1}{5} \\ \hline 6\frac{1}{6} \end{array}=\text{what?}$$ *Ans.* $\frac{2516}{9331}$.

15. $(2\frac{1}{2}\times2\frac{1}{3}+\frac{5}{9}$ of $\frac{7}{16})+(\frac{8}{9})^3\div(7\frac{7}{9}-3\frac{1}{2}\times\frac{49}{50})=$what?

Ans. $\frac{400000}{407511}$.

16. $\left\{\frac{2\frac{7}{11}}{3\frac{17}{33}}\times\frac{4\frac{1}{5}}{7\frac{14}{25}}-\frac{5\frac{11}{20}}{62\frac{7}{16}}\right\}\times4\,\frac{3\frac{11}{17}}{\frac{4\frac{3}{7}}{8}}\div\frac{\frac{1}{20}+\frac{28}{49}-\frac{1}{10}}{4\frac{3}{7}\times5\frac{1}{2}\div200\frac{10}{17}}=$what?

Ans. 3.

17. $2\div2\div2\div2\div2\div2\div2\div2\div\frac{1}{2}\div\frac{1}{2}\div\frac{1}{2}\div\frac{1}{2}\div\frac{1}{2}\div\frac{1}{2}=$what?

Ans. 1.

18. Reduce $\frac{4}{5}$ to thirds. *Ans.* $2\frac{2}{5}$ thirds.

19. What fraction is as much larger than $\frac{5}{6}$ as $\frac{4}{5}$ is less than $\frac{5}{6}$?

Ans. $\frac{13}{15}$.

20. What is the value in the 13th example if a heavy mark be drawn between $\frac{1}{6}$ and $\frac{1}{7}$. *Ans.* $1\frac{3}{4}$.

CHAPTER XI.

CIRCULATING DECIMALS.

I. Change $.\dot{6}\dot{3}$ to a common fraction.

II.
1. $.\dot{6}\dot{3}=.636363+$ etc., *ad infinitum.*
2. $.636363+$etc.,$=.63+.0063+.000063+$etc., *ad infinitum.*
3. This is a geometrical infinite decreasing series whose first term is .63 and ratio $.63\div.0063=\frac{1}{100}$. The sum of such a series is $\frac{a}{1-r}=.63\div(1-\frac{1}{100})=\frac{63}{99}=\frac{7}{11}$.

III. $\therefore .\dot{6}\dot{3}=\frac{7}{11}$.

I. Reduce $\dot{1}.00\dot{1}$ to a common fraction.

II.
1. $\dot{1}.00\dot{1}=1.00110011001100110011+$etc., *ad infinitum.*
2. $\dot{1}.00\dot{1}=1.\dot{0}01\dot{1}$.
3. $1.\dot{0}01\dot{1}=1+.0011+00000011+000000000011+$etc., *ad infinitum.*
4. $.\dot{0}01\dot{1}=$first term.
5. $\frac{1}{10000}=.0011\div00000011=$ratio.
6. $\therefore$ Sum$=\frac{a}{1-r}=.0011\div(1-\frac{1}{10000})=.0011\div\frac{9999}{10000}=\frac{11}{9999}$ $=\frac{1}{909}$.
7. $\therefore 1.\dot{0}01\dot{1}=1+\frac{1}{909}=1\frac{1}{909}$.

(*Ray's H. A., p. 120, ex. 8.*)

III. $\therefore \dot{1}.00\dot{1}=1\frac{1}{909}$.

Remark.—Since the denominator of the ratio is always ten or some power of ten, the numerator of the fraction resulting from subtracting the ratio from 1, will have as many 9's in it as there

are ciphers in the denominator of the ratio. By dividing the first term by this fraction, its numerator becomes the denominator of the fraction required. Hence, a *circulate* may be reduced to a common fraction by writing for the denominator of the repetend as many 9's as there are figures in the repetend. Thus, $.6\dot{3}=.6\frac{3}{9}=.6\frac{1}{3}=\frac{6\frac{1}{3}}{10}=\frac{\frac{19}{3}}{10}=\frac{19}{30}$.

I. Reduce $.034\dot{6}3\dot{9}$ to a common fraction.

1. $.034\dot{6}3\dot{9}=.034\frac{639}{999}=\frac{34\frac{639}{999}}{1000}=\frac{34\times999+639}{\frac{999}{1000}}=\frac{34\times999+639}{1000\times999}$,

$$=\frac{34\times(1000-1)+639}{999000}=\frac{34000-34+639}{999000}=\frac{34000+639-34}{999000}=$$
$$\frac{34639-34}{999000}=\frac{34605}{999000}=\frac{6921}{199800}=\frac{2307}{66600}=\frac{769}{22200}.$$

In case the circulate is mixed, we have the following rule:

1. *For the numerator, subtract that part which precedes the repetend from the whole expression, both quantities being considered as units.*

2. *For the denominator, write as many 9's as there are figures in the repetend, and annex as many ciphers as there are decimal figures before each repetend.*

I. ADDITION OF CIRCULATES.

I. Add $5.0\dot{7}7\dot{0}$, $.\dot{2}\dot{4}$, and $7.\dot{1}2494\dot{3}$.

II. $\begin{cases} 1.\quad 5.0\dot{7}7\dot{0} = 5.0\dot{7}7\dot{0} = 5.0\dot{7}70\dot{7}7\dot{0}7 \text{ etc.} \\ 2.\quad .\dot{2}\dot{4} = .2\dot{4}\dot{2} = .2\dot{4}2424\dot{2}4 \text{ etc.} \\ 3.\quad 7.\dot{1}2494\dot{3} = 7.1\dot{2}4943\dot{1} = 7.1\dot{2}4943\dot{1}2 \text{ etc.} \end{cases}$

III. $\therefore$ Sum$=12.4\dot{4}$ $\qquad 12.4\dot{4}4444\dot{4}$ etc.$=12.4\dot{4}$.

Explanation.—The first thing, in the addition and subtraction of circulates, is to make the circulates *co-originous*, *i. e.*, to make them begin at the same decimal place. That is, if one begins at (say) hundredths, make them all begin at hundredths, providing that each circulate has hundredths repeated. It is best to make them all begin with the circulate whose first repeated figure is farthest from the decimal point, though any order after that may be taken. In the above example we have made them all begin at hundredths. After having made them all begin at hundredths, the next step is to make them conterminous, *i. e.*, to make them all end at the same place. To do this, we find the L. C. M. of the numbers of figures repeated in each circulate, then divide the L. C. M. by the number of figures repeated in each circulate for the number of times the figures as a group must be repeated. Thus, the number of figures in the first repetend is 3; in the second, 2; and in the third, 6.

The L. C. M. of 3, 2, and 6 is 6. $6\div3=2$. $\therefore\ \dot{7}7\dot{0}$ must

be repeated twice. 6÷2=3. ∴ $\dot{4}\dot{2}$ must be repeated three times. 6÷6=1. ∴ $\dot{2}4943\dot{1}$ must be taken once.

I. Add $.94\dot{6}$, $.2\dot{4}\dot{8}$, $5.0\dot{7}7\dot{0}$, $3.\dot{4}88\dot{4}$, and $7.\dot{1}2494\dot{3}$.

II.
1. $.94\dot{6} = .94\dot{6} = .9466666666666\dot{6}$ etc.
2. $.2\dot{4}\dot{8} = .24\dot{8}\dot{4} = .2\dot{4}8484848484\dot{8}48$ etc.
3. $5.0\dot{7}7\dot{0} = 5.07\dot{7}0\dot{7} = 5.07\dot{7}077077077\dot{0}77$ etc.
4. $3.\dot{4}88\dot{4} = 3.48844\dot{8} = 3.4\dot{8}8448844884\dot{4}88$ etc.
5. $7.\dot{1}2494\dot{3} = 7.12\dot{4}9431\dot{2} = 7.12\dot{4}9431249431\dot{2}4$ etc.
6. Sum - - - $=16.8856205620562\dot{0}+$,
$=16.88\dot{5}62\dot{0}$.

III. ∴ Sum$=16.88\dot{5}62\dot{0}$.

II. SUBTRACTION OF CIRCULATES.

I. Subtract $\dot{1}90.47\dot{6}$ from $199.6\dot{4}2857\dot{1}$

II.
1. $199.6\dot{4}2857\dot{1} = 199.6\dot{4}2857\dot{1}4$
2. $\dot{1}90.47\dot{6} = 190.4\dot{7}6190\dot{4}7$
3. Difference $= 9.1\dot{6}6666\dot{6} = 9.1\dot{6}$.

III. ∴ Difference$=9.1\dot{6}$.

I. Subtract $13.\dot{6}3\dot{7}$ from $10\dot{4}.\dot{1}$.

II.
1. $10\dot{4}.\dot{1} = 104.\dot{1}\dot{4} = 104.\dot{1}4141\dot{4}1$ etc.
2. $13.\dot{6}3\dot{7} = 13.\dot{6}3\dot{7} = 13.\dot{6}3763\dot{7}6$ etc.
3. Difference $= 90.\dot{5}0377\dot{6}$

III. ∴ Difference$=90.\dot{5}0377\dot{6}$.

III. MULTIPLICATION OF CIRCULATES.

I. Multiply $.0\dot{7}06\dot{7}$ by $.9\dot{4}3\dot{2}$.

$.0\dot{7}06\dot{7} = .0\dot{7}067\dot{7}$

$.9\dot{4}3\dot{2} = .9\frac{16}{37}$

$.06\dot{3}60\dot{9}$
$.00\dot{3}05\dot{6}$
$.06\dot{6}66\dot{5}$=product.

Multiply by the fraction thus:

$.0\dot{7}067\dot{7}$
16
$.\dot{4}240\dot{6} = .4\dot{2}406\dot{2}$
$.\dot{7}06\dot{7} = .7\dot{0}677\dot{0}$

37)1.1$\dot{3}$083($\dot{3}05\dot{6}$=
1 11
208
185
233
222
11

$.00\dot{3}05\dot{6}$, because the fraction is $.0\frac{16}{37}$

I. Multiply $1.\dot{2}5678\dot{4}$ by $6.420\dot{8}\dot{1}$.

$$1.\dot{2}5678\dot{4}=1.\dot{2}5678\dot{4}2$$
$$6.420\dot{8}\dot{1} = 6.420\tfrac{9}{11}$$

$$\begin{array}{rcl}
.02\dot{5}1356\dot{8} & = & .025\dot{1}3568\dot{5}^{1} \\
.5\dot{0}2713\dot{7} & = & .502\dot{7}1370\dot{2}^{7} \\
7.\dot{5}4070\dot{5} & = & 7.540\dot{7}0554\dot{0}^{7} \\
.001\dot{0}2827\dot{0} & = & .001\dot{0}2827\dot{0}^{0} \\
\hline
 & & 8.069\dot{5}8319\dot{8}
\end{array}$$

Multiply by the fraction thus:

$$\begin{array}{r}
1.\dot{2}5678\dot{4}^{2} \\
.000\frac{9}{11} \\
\hline
11).011\dot{3}1105\dot{7} \\
\hline
.001\dot{0}2827\dot{0}
\end{array}$$

Remark.—In multiplying by any number, begin sufficiently far beyond the last figure of the repetend, so that if there is any to carry it may be added to the repetends of the partial products, making them complete. Thus in the above example, when multiplying by 4, we begin at 5, the second decimal place beyond 4, the last figure of the repetend; and so when we multiply $\dot{4}$ by 4, the first figure of the repetend in the partial product is $\dot{7}$.

IV. DIVISION OF CIRCULATES.

RULE.—*Change the terms to common fractions; then divide as in division of fractions, and reduce the quotient to a repetend.*

I. Divide $.\dot{7}\dot{5}$ by $.\dot{1}$

II. $\begin{cases} 1. & .\dot{7}\dot{5}=\frac{75}{99}=\frac{25}{33}. \\ 2. & .\dot{1}=\frac{1}{9}. \\ 3. & \frac{25}{33}\div\frac{1}{9}=\frac{25}{33}\times 9=\frac{25}{11}=6.8181 \text{ etc.}=6.\dot{8}\dot{1}. \end{cases}$

III. $\therefore .\dot{7}\dot{5}\div .\dot{1}=6.\dot{8}\dot{1}$.

EXAMPLES.

1. Add $.\dot{8}\dot{7}$, $.8$, and $\dot{8}7\dot{6}$. *Ans.* $2.\dot{6}4455\dot{3}$.
2. Add $.\dot{3}$, $.45$, $.\dot{4}\dot{5}$, $.\dot{3}5\dot{1}$, $.6468$, $.646\dot{8}$, $.64\dot{6}\dot{8}$, and $6\dot{4}6\dot{8}$. *Ans.* $4.1766\dot{3}4561\dot{8}$.
3. Add $2\dot{7}.5\dot{6}$, 5.632, $6.\dot{7}$, $16.3\dot{5}\dot{6}$, $.7\dot{1}$, and $6.\dot{1}23\dot{4}$. *Ans.* $63.1\dot{6}9067086888\dot{8}$.
4. Add $5.\dot{1}634\dot{5}$, $8.\dot{6}38\dot{1}$, and $3.\dot{7}\dot{5}$. *Ans.* $17.\dot{5}591912084737409030\dot{2}$.
5. From $31\dot{5}.8\dot{7}$ take $78.\dot{0}37\dot{8}$. *Ans.* $237.\dot{8}3807209549\dot{7}$.
6. From $16.13\dot{4}\dot{7}$ take $11.08\dot{8}\dot{4}$. *Ans.* $5.04\dot{6}\dot{2}$.
7. From $18.16\dot{7}\dot{8}$ take $3.\dot{2}\dot{7}$. *Ans.* $14.89\dot{5}\dot{1}$.
8. From $\frac{9}{17}$ take $\frac{6}{17}$. *Ans.* $.\dot{1}76470588235294\dot{1}$.

9. From $5.\dot{1}234\dot{5}$ take $2.3\dot{5}2345\dot{6}$.
Ans. $2.7\dot{7}1105582166692777798888859999\dot{4}$.

10. Multiply $87.32\dot{5}8\dot{6}$ by 4.37. *Ans.* $381.6140\dot{3}3\dot{8}$.

11. Multiply 382.347 by $.0\dot{3}$. *Ans.* $13.5\dot{1}6953\dot{3}$.

12. Multiply $.\dot{9}62566844919786\dot{0}$ by $.7\dot{5}$. *Ans.* $.\dot{7}\dot{2}$.

13. Divide $234.\dot{6}$ by $.\dot{7}$. *Ans.* $701.\dot{7}1428\dot{5}$.

14. Divide $13.5\dot{1}6953\dot{3}$ by $3.1\dot{4}\dot{5}$. *Ans.* $4.\dot{2}9\dot{7}$.

15. Divide $2.37\dot{0}$ by $4.\dot{9}2307\dot{6}$. *Ans.* $.\dot{4}8\dot{1}$.

16. Divide $.\dot{3}\dot{6}$ by $.2\dot{5}$. *Ans.* $1.\dot{4}22924901185770750988\dot{1}$.

17. Divide $.\dot{7}\dot{2}$ by $.7\dot{5}$. *Ans.* $.\dot{9}62566844919786\dot{0}$.

18. $54.0\dot{6}7813\dot{2} \div 8.\dot{5}9\dot{4} =$ what? *Ans.* $6.\dot{2}9\dot{0}$.

19. $4.9\dot{5}\dot{6} \div .75 =$ what? *Ans.* $6.6\dot{0}8754\dot{2}$.

20. $7.\dot{7}1428\dot{5} \div .\dot{9}5238\dot{0} =$ what? *Ans.* 8.1.

CHAPTER XII.

I. PERCENTAGE AND ITS VARIOUS APPLICATIONS.

1. *Percentage* is a method of computation in which 100 is taken as the basis of comparison.

2. *Per cent.* is from the Latin, *per centum*, *per*, by, and *centum*, a hundred.

3. *The Terms* used in percentage are the *Base*, the *Rate*, the *Percentage*, and the *Amount* or *Difference*.

4. *The Base* is the number on which the percentage is computed.

5. *The Rate* is the number of hundredths of the base which is to be taken.

6. *The Percentage* is the result obtained by taking a certain per cent. of the base.

7. *The Amount* or ***Difference*** is the sum or difference of the base and percentage.

CASE I.

I. What is 10% of \$700?

II. $\begin{cases} 1.\ \ 100\% = \$700. \\ 2.\ \ \ \ 1\% = \frac{1}{100} \text{ of } \$700 = \$7, \text{ and} \\ 3.\ \ \ 10\% = 10 \text{ times } \$7 = \$70. \end{cases}$

III. $\therefore$ 10% of \$700=\$70.

I. What is 8% of \$500?

II. 1. $100\%=\$500$,
2. $1\%=\frac{1}{100}$ of $\$500=\$5$, and
3. $8\%=8$ times $\$5=\$40$.

III. ∴ 8% of $\$500=\$40$.

I. What is $\frac{3}{4}\%$ of 800 men?

II. 1. $100\%=800$ men.
2. $1\%=\frac{1}{100}$ of 800 men=8 men, and
3. $\frac{3}{4}\%=\frac{3}{4}$ times 8 men=6 men.

III. ∴ $\frac{3}{4}\%$ of 800 men=6 men.

I. What is 10% of 20% of \$13.50?

II. (1.) 1. $100\%=\$13.50$.
2. $1\%=\frac{1}{100}$ of $\$13.50=\$.135$, and
3. $20\%=20$ times $\$.135=\$2.70$.
(2.) $100\%=\$2.70$.
(3.) $1\%=\frac{1}{100}$ of $\$2.70=\$.027$, and
(4.) $10\%=10$ times $\$.027=\$.27=27$ cents.

III. ∴ 10% of 20% of $\$13.50=27$ cents.

I. A. had \$1200; he gave 30% to a son, 20% of the remainder to his daughter, and so divided the rest among four brothers that each after the first had \$12 less than the preceding. How much did the last receive?

II. (1.) 1. $100\%=\$1200$,
2. $1\%=\frac{1}{100}$ of $\$1200=\$12$, and
3. $30\%=30$ times $\$12=\$360=$son's share.
4. $\$1200-\$360=\$840=$remainder.
(2.) 1. $100\%=\$840$,
2. $1\%=\frac{1}{100}$ of $\$840=\$8.40$, and
3. $20\%=20$ times $\$8.40=\$168=$daughter's share.
4. $\$840-\$168=\$672=$amount divided among four brothers.
(3.) $100\%=$fourth brother's share;
(4.) $100\%+\$12=$third brother's share.
(5.) $100\%+\$24=$second brother's share, and
(6.) $100\%+\$36=$first brother's share.
(7.) $100\%+(100\%+\$12)+(100\%+\$24)+(100\%+\$36)=400\%+\$72=$am't the four brothers rec'd.
(8.) $\$672=$amount the four brothers received.
(9.) ∴ $400\%+\$72=\$672$.
(10.) $400\%=\$672-\$72=\$600$.
(11.) $1\%=\frac{1}{400}$ of $\$600=\$1.50$.
(12.) $100\%=100$ times $\$1.50=\$150=$fourth brother's share.

III. ∴ The last received \$150. (*R. H. A., p. 191, prob. 25.*)

I. What number increased by 20% of 3.5, diminished by $12\frac{1}{2}\%$ of 9.6, gives $3\frac{1}{2}$?

II. (1.) 100%=the number.
(2.) 1. 100%=3.5,
2. 1%=$\frac{1}{100}$ of 3.5=.035, and
3. 20%=20 times .035=.7.
(3.) 1. 100%=9.6,
2. 1%=$\frac{1}{100}$ of 9.6=.096, and
3. $12\frac{1}{2}\%=12\frac{1}{2}$ times .096=1.2.
(4.) ∴ 100%+.7—1.2=$3\frac{1}{2}$,
(5.) 100%—.5=3.5, and
(6.) 100%=4, the number.

III. ∴ The number=4. (*R. H. A., p. 191, prob. 26.*)

CASE II.

I. $90 is what % of $300?

II. 1. $300=100%,
2. $1=$\frac{1}{300}$ of 100%=$\frac{1}{3}$%, and
3. $90=90 times $\frac{1}{3}$%=30%.

III. ∴ $90 is 30% of $300.

I. 750 men is what % of 12000 men?

II. 1. 12000 men=100%,
2. 1 man=$\frac{1}{12000}$ of 100%=$\frac{1}{120}$%, and
3. 750 men=750 times $\frac{1}{120}$%=$6\frac{1}{4}$%.

III. ∴ 750 men is $6\frac{1}{4}$% of 12000 men.

I. A's money is 50% more than B's; then B's is how many % less than A's?

II. 1. 100%=B.'s money. Then,
2. 100%+50%=150%=A.'s money.
3. 150%=100% of itself.
4. 1%=$\frac{1}{150}$ of 100%=$\frac{2}{3}$%, and
5. 50%=50 times $\frac{2}{3}$%=$33\frac{1}{3}$%.

III. ∴ B.'s money is $33\frac{1}{3}$% less than A.'s

(*R. H. A., p. 192, prob. 11.*)

I. 30% of the whole of an article is how many % of $\frac{2}{3}$ of it?

II. 1. 100%=whole article.
2. $66\frac{2}{3}\%=\frac{2}{3}$ of 100%=$\frac{2}{3}$ of the article.
3. $66\frac{2}{3}$%=100% of itself.
4. $1\%=\dfrac{1}{66\frac{2}{3}}$ of 100%=$1\frac{1}{2}$%, and
5. 30%=30 times $1\frac{1}{2}$%=45%.

III ∴ 30% of the whole of an article is 45% of $\frac{2}{3}$ of it.

(*R. H. A., p. 192, prob. 20.*)

I. If a miller takes 4 quarts for toll from every bushel he grinds, what % does he take for toll?

II.
1. 1 bu.=32 qt.
2. 32 qt.=100%,
3. 1 qt.=$\frac{1}{32}$ of 100%=$3\frac{1}{8}$%, and
4. 4 qt.=4 times $3\frac{1}{8}$%=$12\frac{1}{2}$%.

III. ∴ He takes $12\frac{1}{2}$% for toll.

CASE III.

I. $20 is 5% of what sum?

II.
1. 100%=sum.
2. 5%=$20,
3. 1%=$\frac{1}{5}$ of $20=$4, and
4. 100%=100 times $4=$400.

III. ∴ $20 is 5% of $400.

I. $24 is $\frac{3}{8}$% of what sum?

II.
1. 100%=sum.
2. $\frac{3}{8}$%=$24,
3. $\frac{1}{8}$%=$\frac{1}{3}$ of $24=$8,
4. $\frac{8}{8}$%, or 1%,=8 times $8=$64, and
5. 100%=100 times $64=$6400.

III. ∴ $24 is $\frac{3}{8}$% of $6400.

I. I drew 48% of my funds in bank, to pay a note of $150; how much had I left?

II.
1. 100%=amount in bank.
2. 48%=amount drawn out.
3. 100%—48%=52%=amount left.
4. 48%=$150,
5. 1%=$\frac{1}{48}$ of $150=$3.125, and
6. 52%=52 times $3.125=$162.50=amount left.

III. ∴ $162.50=amount I had left.

I. I pay $13 a month for board, which is 20% of my salary; what is my salary?

II.
1. 100%=my monthly salary.
2. 20%=$13,
3. 1%=$\frac{1}{20}$ of $13=$.65, and
4. 100%=100 times $.65=$65, my monthly salary.
5. ∴ $780=12 times $65=my yearly salary.

III. ∴ My salary=$780. (*R. H. A., p. 194, prob. 20.*)

CASE IV.

I. $540 is 8% greater than what sum?

II.
1. 100%=sum.
2. 100%+8%=108%=sum increased 8%, and
3. $540=sum increased 8%;
4. ∴ 108%=$540.
5. 1%=$\frac{1}{108}$ of $540=$5, and
6. 100%=100 times $5=$500.

III. ∴ $540 is 8% greater than $500.

I. A sold a horse for $150 and gained 25%; what did the horse cost?

II.
1. 100%=cost of horse.
2. 25%=gain.
3. 100%+25%=125%=selling price of horse, and
4. $150=selling price of horse;
5. ∴ 125%=$150,
6. 1%=$\frac{1}{125}$ of $150=$1.20, and
7. 100%=100 times $1.20=$120=cost of horse.

III. ∴ The horse cost $120.

I. I sold two horses for the same price, $150; on one I gained 25% and on the other I lost 25%; what was the cost of each?

II.

A.
1. 100%=cost of first horse.
2. 25%=gain.
3. 100%+25%=125%=selling price of first horse,
4. $150=selling price of first horse;
5. ∴ 125%=$150,
6. 1%=$\frac{1}{125}$ of $150=$1.20, and
7. 100%=100 times $1.20=$120=cost of first horse.

B.
1. 100%=cost of second horse.
2. 25%=loss on second horse.
3. 100%—25%=75%=selling price of 2d horse, and
4. $150=selling price of second horse;
5. ∴ 75%=$150,
6. 1%=$\frac{1}{75}$ of $150=$2, and
7. 100%=100 times $2=$200=cost of second horse.

III. ∴ $120=cost of first horse, and
$200=cost of second horse.

I. A coat cost $32; the trimmings cost 70% less, and the making 50% less than the cloth; what did each cost?

II.
1. 100%=cost of cloth. Then
2. $100\%-70\%=30\%$=cost of trimmings, and
3. $100\%-50\%=50\%$=cost of making.
4. $100\%+30\%+50\%=180\%$=cost of coat.
5. $32=cost of coat;
6. $\therefore 180\%=\$32$,
7. $1\%=\frac{1}{180}$ of $\$32=\$.1777\frac{7}{9}$.
8. $100\%=100$ times $\$.1777\frac{7}{9}=\$17.77\frac{7}{9}$=cost of cloth.
9. $30\%=30$ times $\$.1777\frac{7}{9}=\$5.33\frac{1}{3}$=cost of trimming.
10. $50\%=50$ times $\$.1777\frac{7}{9}=\$8.88\frac{8}{9}$=cost of making.

III. $\therefore$ $17.77\frac{7}{9}$=cost of cloth,
$ 5.33\frac{1}{3}$=cost of trimmings, and
$ 8.88\frac{8}{9}$=cost of making.

(*R. H. A., p. 196, prob. 12.*)

I. In a company of 87, the children are $37\frac{1}{2}\%$ of the women, who are $44\frac{4}{9}\%$ of the men; how many of each?

II.
(1.) 100%=number of men. Then
(2.) $44\frac{4}{9}\%$=number of women.
(3.)
1. $100\%=44\frac{4}{9}\%$,
2. $1\%=\frac{1}{100}$ of $44\frac{4}{9}\%=.44\frac{4}{9}\%$, and
3. $37\frac{1}{2}\%=37\frac{1}{2}$ times $.44\frac{4}{9}\%=16\frac{2}{3}\%$=number of children in terms of the number of men.

(4.) $100\%+44\frac{4}{9}\%+16\frac{2}{3}\%=161\frac{1}{9}\%$=number in the company,
(5.) 87=number in the company;
(6.) $\therefore 161\frac{1}{9}\%=87$,
(7.) $1\%=\dfrac{1}{161\frac{1}{9}}$ of $87=.54$,
(8.) $100\%=100$ times $.54=54$=number of men,
(9.) $44\frac{4}{9}\%=44\frac{4}{9}$ times $.54=24$=number of women, and
(10.) $16\frac{2}{3}\%=16\frac{2}{3}$ times $.54=18$=number of children.

III. $\therefore$ 54=number of men,
24=number of women, and
18=number of children.

(*R. H. A., p. 197, prob. 20.*)

I. Our stock decreased $33\frac{1}{3}\%$, and again 20%; then it rose 20%, and again $33\frac{1}{3}\%$; we have thus lost $66; what was the stock at first?

II.
(1.) 100%=original stock.
(2.) $33\frac{1}{3}\%$=decrease.
(3.) $100\%-33\frac{1}{3}\%=66\frac{2}{3}\%$=stock after first decrease.
(4.) 1. $100\%=66\frac{2}{3}\%$,
2. $1\%=\frac{1}{100}$ of $66\frac{2}{3}\%=\frac{2}{3}\%$, and
3. $20\%=20$ times $\frac{2}{3}\%=13\frac{1}{3}\%$=second decrease.
4. $66\frac{2}{3}\%-13\frac{1}{3}\%=53\frac{1}{3}\%$=stock after second decrease.
(5.) 1. $100\%=53\frac{1}{3}\%$,
2. $1\%=\frac{1}{100}$ of $53\frac{1}{3}\%=.53\frac{1}{3}\%$, and
3. $20\%=20$ times $.53\frac{1}{3}\%=10\frac{2}{3}\%$=first increase.
4. $53\frac{1}{3}\%+10\frac{2}{3}\%=64\%$=stock after first increase
(6.) 1. $100\%=64\%$,
2. $1\%=\frac{1}{100}$ of $64\%=.64\%$, and
3. $33\frac{1}{3}\%=33\frac{1}{3}$ times $.64\%=21\frac{1}{3}\%$=second increase.
4. $64\%+21\frac{1}{3}\%=85\frac{1}{3}\%$=stock after second increase.
(7.) $100\%-85\frac{1}{3}\%=14\frac{2}{3}\%$=whole loss;
(8.) $66=whole loss;
(9.) $\therefore 14\frac{2}{3}\%=\$66$;
(10.) $1\%=\frac{1}{14\frac{2}{3}}$ of $\$66=\$4.50$, and
(11.) $100\%=100$ times $\$4.50=\$450$=original stock.

III. $\therefore$ $450=original stock.

I. A brewery is worth 4% less than a tannery, and the tannery 16% more than the boat; the owner of the boat has traded it for 75% of the brewery, losing thus $103; what is the tannery worth?

FIRST SOLUTION.

II.
(1.) 100%=value of the tannery. Then
(2.) $100\%-4\%=96\%$=value of the brewery.
(3.) 1. 100%=value of the boat. Then
2. $100\%+16\%=116\%$=value of tannery in terms of [the boat.
3. $116\%=100\%$, the value of tannery from step (1),
4. $1\%=\frac{1}{116}$ of $100\%=\frac{25}{29}\%$, and
5. $100\%=100$ times $\frac{25}{29}\%=86\frac{6}{29}\%$=value of the boat in terms of the tannery.
(4.) 1. $100\%=96\%$,
2. $1\%=\frac{1}{100}$ of $96\%=.96\%$, and
3. $75\%=75$ times $.96\%=72\%$=what the owner of the boat received for it.
(5.) $\therefore 86\frac{6}{29}\%-72\%=14\frac{6}{29}\%$=what the owner of the boat lost in the trade.
(6.) $103=what he lost;
(7.) $\therefore 14\frac{6}{29}\%=\$103$,
(8.) $1\%=\frac{1}{14\frac{6}{29}}$ of $\$103=\$7.25$, and
(9.) $100\%=100$ times $\$7.25=\$725$=value of tannery.

III. $\therefore$ $725=value of the tannery. (*R. H. A., p. 197, prob. 23.*)

Remark.—The value of the brewery and boat being expressed in terms of the tannery, 75% of the brewery is also expressed in terms of the tannery; hence, it is plain that the owner of the boat has traded $86\frac{6}{29}\%$ for 72% of the same value, losing $86\frac{6}{29}\%-72\%$, or $14\frac{6}{29}\%$.

SECOND SOLUTION.

II.
- (1.) 100%=value of the boat. Then
- (2.) 100%+16%=116%=value of the tannery.
- (3.)
  1. 100%=116%,
  2. $1\%=\frac{1}{100}$ of 116%=1.16%, and
  3. 4%=4 times 1.16%=4.64%.
- (4.) 116%—4.64%=111.36%=the value of brewery in terms of the boat.
- (5.)
  1. 100%=111.36%,
  2. $1\%=\frac{1}{100}$ of 111.36%=1.1136%, and
  3. 75%=75 times 1.1136%=83.52%=what the owner of the boat received for it.
- (6.) ∴ 100%—83.52%=16.48%=what he lost in the trade.
- (7.) \$103=what he lost.
- (8.) ∴ 16.48%=\$103,
- (9.) $1\%=\frac{1}{16.48}$ of \$103=\$6.25, and
- (10.) 116%=116 times \$6.25=\$725=value of tannery.

III. ∴ \$725=value of the tannery.

THIRD SOLUTION.

II.
- (1.) 100%=value of brewery.
- (2.)
  1. 100%=value of tannery. Then
  2. 100%—4%=96%=value of the tannery.
  3. ∴ 96%=100%, the value of brewery in step (1),
  4. $1\%=\frac{1}{96}$ of $100\%=1.04\frac{1}{6}\%$, and
  5. 100%=100 times $1.04\frac{1}{6}\%=104\frac{1}{6}\%$=value the tannery in terms of the brewery.
- (3.)
  1. 100%=value of boat. Then
  2. 100%+16%=116%=value of the tannery in terms of the boat.
  3. ∴ $116\%=104\frac{1}{6}\%$, the value of the tannery in step 5 of (2),
  4. $1\%=\frac{1}{116}$ of $104\frac{1}{6}\%=.89\frac{139}{174}\%$, and
  5. 100%=100 times $.89\frac{139}{174}\%=89\frac{139}{174}\%$=value of the boat in terms of the tannery, and consequently in terms of the brewery.
- (4.) ∴ $89\frac{139}{174}\%-75\%=14\frac{139}{174}\%$=what the owner of the boat lost in the trade.
- (5.) \$103=what the owner of the boat lost;
- (6.) ∴ $14\frac{139}{174}\%$=\$103,
- (7.) $1\%=\dfrac{1}{14\frac{139}{174}}$ of \$103=\$6.96, and
- (8.) $104\frac{1}{6}\%=104\frac{1}{6}$ times \$6.96=\$725=value of tannery.

III. ∴ \$725=value of of the tannery.

Remark.—In step 5 of (3), we have the value of the boat in terms of the tannery; but the value of the tannery is in terms of the brewery: hence, the value of the boat is also in terms of the brewery. The owner of the boat, therefore, traded $89\frac{139}{174}\%$ for 75% of the same value.

MISCELLANEOUS PROBLEMS.

I. A man sold a horse for \$175, which was $12\frac{1}{2}\%$ less than the horse cost; what did the horse cost?

II.
1. 100%=cost of horse.
2. $12\frac{1}{2}\%$=loss.
3. $100\%-12\frac{1}{2}\%=87\frac{1}{2}\%$=selling price.
4. \$175=selling price.
5. ∴ $87\frac{1}{2}\%$=\$175,
6. $1\%=\frac{1}{87\frac{1}{2}}$ of \$175=\$2, and
7. 100%=100 times \$2=\$200,

III. ∴ \$200=cost of the horse. (*R. 3d p., p. 204, prob. 5.*)

I. A miller takes for toll 6 quarts from every 5 bushels of wheat ground; what % does he take for toll.?

II.
1. 1 bu.=32 qt.
2. 5 bu.=5 times 32 qt.=160 qt.
4. 160 qt.=100%,
4. 1 qt.=$\frac{1}{160}$ of $100\%=\frac{5}{8}\%$, and
5. 6 qt.=6 times $\frac{5}{8}\%=3\frac{3}{4}\%$.

III. ∴ He takes $3\frac{3}{4}\%$ for toll. (*R. 3d p., p. 204, prob. 11.*)

I. A farmer owning 45% of a tract of land, sold 540 acres, which was 60% ot what he owned; how many acres were there in the tract?

II.
(1.) 100%=number of acres in the tract.
(2.)
1. 100%=numbers of acres the farmer owned.
2. 60%=number of acres the farmer sold.
3. 540 acres=what he sold.
4. ∴ 60%=540 acres,
5. $1\%=\frac{1}{60}$ of 540 acres=9 acres, and
6. 100%=100 times 9 acres=900 acres=what he owned.

(3.) 45%=what he owned.
(4.) ∴ 45%=900 acres,
(5.) $1\%=\frac{1}{45}$ of 900 acres=20 acres, and
(6.) 100%=100 times 20 acres=2000=number of acres in the tract.

III. ∴ The tract contained 2000 acres.

(*R. 3d p., p. 204, prob. 12.*)

I. A, wishing to sell a cow and a horse to B, asked 150% more for the horse than for the cow; he then reduced the price of the cow 25%, and the horse $33\frac{1}{3}$%, at which price B took them, paying \$290; what was the price of each?

II.
(1.) 100%=asking price of the cow. Then
(2.) 100%+150%=250%=asking price of the horse.
(3.) 100%—25%=75%=selling price of cow.
(4.) 1. 100%=250%,
2. 1%=$\frac{1}{100}$ of 250%=2.50%, and
3. $33\frac{1}{3}$%=$33\frac{1}{3}$ times 2.5%=$83\frac{1}{3}$%=reduction on the asking price of the horse.
(5.) 250%—$83\frac{1}{3}$%=$166\frac{2}{3}$%=selling price of the horse.
(6.) 75%+$166\frac{2}{3}$%=$241\frac{2}{3}$%=selling price of both.
(7.) \$290=selling price of both.
(8.) ∴ $241\frac{2}{3}$%=\$290,
(9.) 1%=$\frac{1}{241\frac{2}{3}}$ of \$290=\$1.20, and
(10.) 75%=75 times\$\$1.20=\$90=selling price of the cow.
(11.) $166\frac{2}{3}$%=$166\frac{2}{3}$ times \$1.20=\$200=selling price of the horse.

III. ∴ \$ 90=selling price of the cow, and
\$200=selling price of the horse.

(*Brooks' H. A., p. 243, prob. 18.*)

I. A mechanic contracts to supply dressed stone for a church for \$87560, if the rough stone cost him 18 cents a cubic foot; but if he can get it for 16 cents a cubic foot, he will deduct 5% from his bill; required the number of cubic feet and the charge for dressing the stone.

II.
1. 100%=\$87560.
2. 1%=$\frac{1}{100}$ of \$87560=\$875.60, and
3. 5%=5 times \$875.60=\$4378=the deduction.
4. 18¢—16¢=2¢=the deduction per cubic foot.
5. ∴ \$4378=the deduction of 4378÷.02, or 218900 cubic feet. Then
6. \$87560=cost of 218900 cubic feet.
7. \$.40=\$87560÷218900=cost of one cubic feet.
8. ∴ \$.40—\$.18=\$.22=cost of dressing per cubic foot.

III. ∴ 218900=number of cubic feet, and
22 cents=cost of dressing per cubic foot.

(*Brooks' H. A., p. 241, prob. 21.*)

EXAMPLES.

1. A merchant, having \$1728 in the Union Bank, wishes to withdraw 15%; how much will remain? *Ans.* \$1468.80.

2. A Colonel whose regiment consisted of 900 men, lost 8% of them in battle, and 50% of the remainder by sickness; how many had he left? *Ans.* 414 men.

3. What % of \$150 is 25% of \$36? *Ans.* 6%.

4. What % of $\frac{1}{2}$ of $\frac{4}{5}$ of $\frac{8}{3}$ is $\frac{1}{3}$? *Ans.* $31\frac{1}{4}$%.

5. If a man owning 45% of a mill, should sell $33\frac{1}{3}$% of his share for \$450; what would be the value of the mill? *Ans.* \$3000.

6. A. expends in a week \$24, which exceeds by $33\frac{1}{3}$% his earnings in the same time. What were his earnings? *Ans.* \$18.

7. Bought a carriage for \$123.06, which was 16% less than I paid for a horse; what did I pay for the horse? *Ans.* \$146.50.

8. Bought a horse, buggy, and harness for \$500. The horse cost $37\frac{1}{2}$% less than the buggy, and the harness cost 70% less than the horse; what was the price of each?
Ans. buggy $\$275\frac{25}{29}$, horse $\$172\frac{12}{29}$, and harness $\$51\frac{21}{29}$.

9. I have 20 yards of yard-wide cloth, which will shrink on sponging 4% in length and 5% in width; how much less than 20 square yards will there be after sponging? *Ans.* $1\frac{19}{25}$ yards.

10. A. found \$5; what was his gain %? *Ans.* ∞.

11. The population of a city whose gain of inhabitants in 5 years has been 25%, is 87500; what was it 5 years ago? *Ans.* 70000.

12. The square root of 2 is what % of the square root of 3? *Ans.* $\sqrt{6} \times 100$%.

13. A laborer had his wages twice reduced 10%; what did he receive before the reduction, if he now receives $\$2.02\frac{1}{2}$ per day? *Ans.* \$2.50.

14. The cube root of 2985984 is what % of the square root of the same number? *Ans.* $8\frac{1}{3}$%.

15. A man sold two horses for the same price \$210; on one he gained 25%, and on the other he lost 25%; how much did he gain, supposing the second horse cost him $\frac{2}{3}$ as much as the first? *Ans.* \$10.

16. A merchant sold goods at 20% gain, but had it cost him $49 more he would have lost 15% by selling at the same price; what did the goods cost him? *Ans.* $119.

17. If an article had cost 20% more, the gain would have been 25% less; what was the gain %? *Ans.* 50%.

II. COMMISSION.

1. ***Commission*** is the percentage paid to an agent for the transaction of business. It is computed on the actual amount of the sale.

2. ***An Agent, Factor,*** or ***Commission Merchant,*** is a person who transacts business for another.

3. ***The Net Proceeds*** is the sum left after the commission and charges have been deducted from the amount of the sales or collections.

4. ***The Entire Cost*** is the sum obtained by adding the commission and charges to the amount of a purchase.

I. An agent received $210 with which to buy goods; after deducting his commission of 5%, what sum must he expend?

II.
1. 100%=what he must expend.
2. 5%=his commission.
3. 100%+5%=105%=what he receives.
4. $210=what he receives.
5. ∴ 105%=$210,
6. 1%=$\frac{1}{105}$ of $210=$2, and
7. 100%=100 times $2=$200=what he expends.

III. ∴ $200=what he must expend.

(*R. 3d p., p. 207, prob. 4.*)

Note.—Since the agent's commission is in the $210, we must not take 5% of $210; for we would be computing commission on his commission. Thus, 5% of ($200+$10)=$10+$.50. This is $.50 too much.

I. An agent sold my corn, and after reserving his commission, invested the proceeds in corn at the same price; his commission, buying and selling was 3%, and his whole charge $12; for what was the corn first sold?

II.
(1.) 100%=cost of the corn.
(2.) 3%=the commission.
(3.) $100\%-3\%=97\%$=net proceeds, which he invested in corn.
(4.)
1. 100%=cost of second lot of corn.
2. 3%=the commission.
3. $100\%+3\%=103\%$=entire cost of second lot of corn.
4. 97%=entire cost of second lot of corn.
5. $\therefore 103\%=97\%$,
6. $1\%=\frac{1}{103}$ of $97\%=\frac{97}{103}\%$, and
7. $100\%=100$ times $\frac{97}{103}\%=94\frac{18}{103}\%$=cost of second lot of corn in terms of the first.
8. $3\%=3$ times $\frac{97}{103}\%=2\frac{85}{103}\%$ = commission on second lot.

(5.) $3\%+2\frac{85}{103}\%=5\frac{85}{103}\%$=whole commission.
(6.) \$12=whole commission.
(7.) $\therefore 5\frac{85}{103}\%=\$12$,
(8.) $1\%=\frac{1}{5\frac{85}{103}}$ of $\$12=\$2.06$, and
(9.) $100\%=100$ times $\$2.06=\$206$=cost of first lot of corn

III. $\therefore$ \$206=cost of first lot of corn. (*R. H. A., p. 219, prob. 10.*)

I. Sold cotton on commission, at 5%; invested the net proceeds in sugar, commission, 2%; my whole commission was \$210; what was the value of the cotton and sugar?

II.
(1.) 100%=cost of cotton.
(2.) 5%=commission.
(3.) $100\%-5\%=95\%$=net proceeds, which he invested in sugar.
1. 100%=cost of sugar.
2. 2%=commission.
3. 102%=entire cost of sugar.
4. 95%=entire cost of sugar.
5. $\therefore 102\%=95\%$,
6. $1\%=\frac{1}{102}$ of $95\%=\frac{95}{102}\%$, and
7. $100\%=100$ times $\frac{95}{102}\%=93\frac{7}{51}\%$=cost of sugar in terms of cotton.
8. $2\%=2$ times $\frac{95}{102}\%=1\frac{44}{51}\%$=commission on the sugar.

(4.) $5\%+1\frac{44}{51}\%=6\frac{44}{51}\%$=whole commission.
(5.) \$210=whole commission.
(6.) $\therefore 6\frac{44}{51}\%=\$210$,
(7.) $1\%=\frac{1}{6\frac{44}{51}}$ of $\$210=\$30.60$, and
(8.) $100\%=100$ times $\$30.60=\$3060$=cost of cotton.
(9.) $93\frac{7}{51}\%=93\frac{7}{51}$ times $\$30.60=\$2850$=cost of sugar.

III. $\therefore$ \$3060=cost of cotton, and
\$2850=cost of sugar. (*R. H. A., p. 219, prob. 6.*)

I. A lawyer received \$11.25 for collecting a debt; his commission being 5%; what was the amount of the debt?

II.
1. 100%=amount of the debt.
2. 5%=commission.
4. \$11.25=commission.
4. ∴ 5%=\$11.25.
5. $1\% = \frac{1}{5}$ of \$11.25=\$2.25, and
6. 100%=100 times \$2.25=\$225=amount of the debt.

III. ∴ \$225=amount of debt.

(*R, 3d p., p. 207, prob. 6.*)

I. Charge \$52.50 for collecting a debt of \$525; what was the rate of commission?

II.
1. \$525=100%.
2. $\$1 = \frac{1}{525}$ of $100\% = \frac{4}{21}\%$, and.
3. \$52.50=52.5 times $\frac{4}{21}\%$=10%=rate of commission.

III. ∴ 10%=rate of commission.

I. My agent sold my flour at 4% commission; increasing the proceeds by \$4.20, I ordered the purchase of wheat at 2% commission; after which, wheat declining $3\frac{1}{3}\%$, my whole loss was \$5; what was the flour worth?

II.
(1.) 100%=cost of flour.
(2.) 4%=commission on flour.
(3.) 100%—4%=96%=net proceeds.
(4.)
1. 100%=cost of wheat.
2. 2%=commission on wheat.
3. 100%+2%=102%=entire cost of wheat.
4. 96%+\$4.20=entire cost of wheat.
5. ∴ 102%=96%+\$4.20,
6. $1\% = \frac{1}{102}$ of $(96\% + \$4.20) = .94\frac{2}{17}\% + \$.0411\frac{13}{17}$,
7. $100\% = 100$ times $(.94\frac{2}{17}\% + \$.0411\frac{13}{17}) = 94\frac{2}{17}\% + \$4.11\frac{13}{17}$=cost of wheat.
8. $2\% = 2$ times $(.94\frac{2}{17}\% + \$.0411\frac{13}{17}) = 1\frac{15}{17}\% + \$.08\frac{4}{17}$ =commission on wheat.

(5.)
1. $100\% = 94\frac{2}{17}\% + \$4.11\frac{13}{17}$,
2. $1\% = \frac{16}{17}\% + \$.04\frac{2}{17}$, and
3. $3\frac{1}{3}\% = 3\frac{1}{3}$ times $(\frac{16}{17}\% + \$.04\frac{2}{17}) = 3\frac{7}{51}\% + \$.13\frac{37}{51}$= loss on wheat.

(6.) $4\% + 1\frac{15}{17}\% + \$.08\frac{4}{17} + 3\frac{7}{51}\% + \$.13\frac{37}{51} = 9\frac{1}{51}\% + \$.21\frac{49}{51}$=whole loss.
(7.) \$5=whole loss.
(8.) ∴ $9\frac{1}{51}\% + \$.21\frac{49}{51} = \$5$, or
(9.) $9\frac{1}{51}\% = \$5 - \$.21\frac{49}{51} = \$4.78\frac{2}{51}$.
(10.) $1\% = \dfrac{1}{9\frac{1}{51}}$ of $\$4.78\frac{2}{51} = \$.53$, and
(11.) 100%=100 times \$.53=\$53.

III. ∴ \$53=cost of flour. (*R. H. A., p. 219, prob. 11.*)

EXAMPLES.

1. A broker in New York exchanged \$4056 on Canal Bank, Portland, at $\frac{5}{8}\%$; what did he receive for his trouble?

Ans. \$25.35.

2. A sold on commission for B 230 yards of cloth at \$1.25 per yard, for which he received a commission of $3\frac{1}{2}\%$; what was his commission and what sum did he remit?

Ans. Commission \$10.06$\frac{1}{4}$, and Remittance \$277.43$\frac{3}{4}$.

3. A sold a lot of books on commission of 20%, and remitted \$160; for what were the books sold? *Ans.* \$200.

4. A lawyer charged \$80 for collecting \$200; what was his rate of commission? *Ans.* 40%

5. I sent my agent \$1364.76 to be invested in pork at \$6 per bbl. after deducting his commission of 2%; how many barrels of pork did he buy? *Ans.* 223 bbl.

6. How much money must I send my agent, so that he may purchase 250 bbl. of flour for me at \$6.25 per bbl., if I pay him $2\frac{1}{2}\%$ commission? *Ans.* \$1601.5625.

7. If an agent's commission was \$200, and his rate of commission 5%; what amount did he invest? *Ans.* \$4000

8. My agent sold cattle at 10% commission, and after I increased the proceeds by \$18, I ordered him to buy hogs at 20% commission. The hogs had declined $6\frac{2}{3}\%$, when he sold them at $14\frac{2}{7}\%$ commission. I lost in all \$68; what did the cattle sell for?

Ans. \$200.

9. An agent sells flour on commission of 2%, and purchases goods on true commission of 3%. If he had received 3% for selling and 2% for buying, his whole commission would have been \$5 more. Find the value of the goods bought.

Ans. \$10506.

III. TRADE DISCOUNT.

1. ***Trade Discount*** is the discount allowed in the purchas and sale of merchandise.

2. ***A List,*** or ***Regular Price,*** is an established price, assumed by the seller as a basis upon which to calculate discount.

3. ***A Net Price*** is a fixed price from which no discount is allowed.

4. ***The Discount*** is the deduction from the list, or regular price.

I. Sold 20 doz. feather dusters, giving the purchaser a discount of 10, 10 and 10% off, his discounts amounting to \$325.20; how much was my price per dozen?

II.
- (1.) 100%=whosesale price.
- (2.) 10% of 100%=10%=first discount.
- (3.) 100%—10%=90%=first net proceeds.
- (4.)
  1. 100%=90%,
  2. 1%=$\frac{1}{100}$ of 90%=$\frac{9}{10}$%, and
  3. 10%=10 times $\frac{9}{10}$%=9%=second discount.
  4. 90%—9%=81%=second net proceeds.
- (5.)
  1. 100%=81%,
  2. 1%=$\frac{1}{100}$ of 81%=$\frac{81}{100}$%, and
  3. 10%=10 times $\frac{81}{100}$%=8.1%=third discount.
- (6.) 10%+9%+8.1%=27.1%=sum of discounts.
- (7.) \$325.20=sum of discounts.
- (8.) ∴ 27.1%=\$325.20,
- (9.) 1%=$\frac{1}{27.1}$ of \$325.20=\$12, and
- (10.) 100%=100 times \$12=\$1200=wholesale price of 20 dozen.
- (11.) \$60=\$1200÷20=wholesale price of 1 dozen.

III. ∴ \$60=wholesale price per dozen.

(*R. 3d p., p. 209, prob. 5.*)

I. Bought 100 dozen stay bindings at 60 cents per dozen for 40, 10, and 7½% off; what did I pay for them?

II.
- (1.) 60¢=list price of 1 dozen.
- (2.) \$60=100 times \$.60=list price of 100 dozen.
- (3.)
  1. 100%=\$60,
  2. 1%=$\frac{1}{100}$ of\$60=\$.60, and
  3. 40%=40 times \$.60=\$24=first discount.
  4. \$60—\$24=\$36=first net proceeds.
- (4.)
  1. 100%=\$36,
  2. 1%=$\frac{1}{100}$ of \$36=\$.36. and
  3. 10%=10 times \$.36=\$3.60=second discount.
  4. \$36—\$3.60=\$32.40=second net proceeds.
- (5.)
  1. 100%=\$32.40,
  2. 1%=$\frac{1}{100}$ of \$32.40=\$.324, and
  3. 7½%=7½ times \$.324=\$2.43=third discount.
  4. \$32.40—\$2.43=\$29.97=cost.

III. ∴ I paid \$29.97. (*R. 3d p., p. 209, prob. 6.*)

I. A retail dealer buys a case of slates containing 10 dozen for \$50 list, and gets 50, 10, and 10% off; paying for them in the usual time, he gets an additional 2%; what did he pay per dozen for the slates?

II. (1.)
1. 100%=\$50.
2. 1%=$\frac{1}{100}$ of \$50=\$.50.
3. 50%=50 times \$.50=\$25=first discount.
4. \$50—\$25=\$25=first net proceeds.

(2.)
1. 100%=\$25.
2. 1%=$\frac{1}{100}$ of \$25=\$.25.
3. 10%=10 times \$.25=\$2.50=second discount
4. \$25—\$2.50=\$22.50=second net proceeds.

(4.)
1. 100%=\$22.50.
2. 1%=$\frac{1}{100}$ of \$22.50=\$.225.
3. 10%=10 times \$.225=\$2.25=third discount.
4. \$22.50—\$2.25=\$20.25=third net proceeds.

(5.)
1. 100%=\$20.25.
2. 1%=$\frac{1}{100}$ of \$20.25=\$.2025.
3. 2%=2 times \$.2025=\$.405=fourth discount.
4. \$20.25—\$.405=\$19.845=cost of 10 dozen slates.
5. \$1.9845=\$19.845÷10=cost of 1 dozen slates.

III. ∴ \$1.9845=cost of 1 dozen slates.

(R. 3d p., p. 209, prob. 9.)

I. Sold a case of hats containing 3 dozen, on which I had received a discount of 10% and made a profit of $12\frac{1}{2}$% or $37\frac{1}{2}$¢ on each hat; what was the wholesale merchant's price per case?

II.
(1.) $37\frac{1}{2}$¢=profit on one hat.
(2.) \$13.50=36 times \$.$37\frac{1}{2}$=profit on 3 dozen hats.
(3.) 100%=wholesale merchant's price per case.
(4.) 10%=discount.
(5.) 100%—10%=90%=my cost.
(6.)
1. 100%=90%.
2. 1%=$\frac{1}{100}$ of 90%=.9%.
3. $12\frac{1}{2}$%=$12\frac{1}{2}$ times .9%=$11\frac{1}{4}$%=profit in terms of wholesale price.

(7.) ∴ $11\frac{1}{4}$%=\$13.50.
(8.) 1%=$\frac{1}{11\frac{1}{4}}$ of \$13.50=\$1.20.
(9.) 100%=100 times \$1.20=\$120=wholesale merchant's price per case.

III. ∴ \$120=wholesale merchant's price per case.

(R. 3d p., p. 212, prob. 4.)

I. A bookseller purchased books from the publishers at 20% off the list; if he retail them at the list what will be his per cent. of profit?

II.
1. 100%=list price.
2. 20%=discount.
3. $100\%-20\%=80\%$=cost.
4. 100%=bookseller's selling price, because he sold them at the list price.
5. $\therefore 100\%-80\%=20\%$=gain.
6. $80\%=100\%$ of itself.
7. $1\%=\frac{1}{80}$ of $100\%=1\frac{1}{4}\%$, and
8. $20\%=20$ times $1\frac{1}{4}\%=25\%$=his gain $\%$.

III. $\therefore 25\%$=his $\%$ of profit. (*R. 3d p., p. 211, prob. 1.*)

Note.—Observe that since his cost is 80%, and his gain 20%, we wish to know what $\%$ 20% is of 80%. It will become evident if we suppose the list price to be (say) $400, and then proceed to find the $\%$ of gain as in the above solution.

I. Bought 50 gross of rubber buttons for 25, 10, and 5% off; disposed of the lot for $35.91, at a profit of 12%; what was the list price of the buttons per gross?

II.
(1.) 100%=list price.
(2.) 25% of $100\%=25\%$=first discount.
(3.) $100\%-25\%=75\%$=first net proceeds.
(4.)
1. $100\%=75\%$,
2. $1\%=\frac{1}{100}$ of $75\%=\frac{3}{4}\%$, and
3. $10\%=10$ times $\frac{3}{4}\%=7\frac{1}{2}\%$.
4. $75\%-7\frac{1}{2}\%=67\frac{1}{2}\%$=second net proceeds
(5.)
1. $100\%=67\frac{1}{2}\%$,
2. $1\%=\frac{1}{100}$ of $67\frac{1}{2}\%=.67\frac{1}{2}\%$, and
3. $5\%=5$ times $.67\frac{1}{2}\%=3.375\%$=third discount.
4. $67\frac{1}{2}\%-3.375\%=64.125\%$=cost.
(6.)
1. $100\%=64.125\%$,
2. $1\%=.64125\%$, and
3. $12\%=12$ times $.64125\%=7.695\%$=gain.
4. $\therefore 64.125\%+7.695\%=71.82\%$=selling price.
(7.) $35.91=selling price.
(8.) $\therefore 71.82\%=\$35.91$,
(9.) $1\%=\frac{1}{71.82}$ of $\$35.91=\$.50$, and
(10.) $100\%=100$ times $\$.50=\$50$=list price of 50 gross,
(11.) $\$1.00=\$50\div50$=list price of one gross.

III. $\therefore$ $1.00=list price of one gross.
(*R. 3d p., p. 212, prob. 10.*)

I A dealer in notions buys 60 gross shoestrings at 70¢ per gross, list, 50, 10, and 5% off; if he sell them at 20, 10, and 5% off list, what will be his profit?

II.
- (1.) 70¢=list price of one gross.
- (2.) $42=60 times $.70=list price of 60 gross.
- (3.)
  1. 100%=$42.
  2. 1%=$\frac{1}{100}$ of $42=$.42.
  3. 50%=50 times $.42=$21=first discount.
  4. $42—$21=$21=first net proceeds.
- (4.)
  1. 100%=$21.
  2. 1%=$\frac{1}{100}$ of $21=$.21.
  3. 10%=10 times $.21=$2.10=second discount.
  4. $21—$2.10=$18.90=second net proceeds.
- (5.)
  1. 100%=$18.90.
  2. 1%=$.189.
  3. 5%=$.945=third discount.
  4. $18.90—$.945=$17.955=cost.
- (6.)
  1. 100%=$42.
  2. 1%=$\frac{1}{100}$ of $42=$.42.
  3. 20%=20 times $.42=$8.40=first conditional discount.
  4. $42—$8.40=$33.60=first conditional net proceeds.
- (7.)
  1. 100%=$33.60.
  2. 1%=$\frac{1}{100}$ of $33.60=$.336.
  3. 10%=10 times $.336=$3.36=second conditional discount.
  4. $33.60—$3.36=$30.24=second conditional net proceeds.
- (8.)
  1. 100%=$30.24.
  2. 1%=$\frac{1}{100}$ of $30.24=$.3024.
  3. 5%=5 times $.3024=$1.512=third conditional discount.
  4. $30.24—$1.512=$28.728=selling price.
- (9.) ∴ $28.728—$17.955=$10.773=his profit.

III. ∴ $10.773=his profit (*R. 3d p., p. 212, prob. 9.*)

EXAMPLES.

1. Bought a case of slates containing 12 doz. for $80 list, and got 45, 10, and 10% off; getting an additional 2% off for prompt payment, what did I pay per dozen for the slates? *Ans.* $3.1752.

2. Bought a case of hats containing 4 doz., on which I received a discount of 40, 20, 10, 5, and $2\frac{1}{2}$% off. If I sell them at $4 a piece making a profit of 20%, what is the wholesale merchant's price per case? *Ans.* $$409\frac{7683}{13013}$.

3. If I receive a discount of 20, 10, and 5% off, and sell at a discount of 10, 5, and $2\frac{1}{2}$% off; what is my % of gain? *Ans.* $21\frac{2}{5}$%—.

4. A bill of goods amounted to $2400; 20% off being allowed, what was paid for the goods? *Ans.* $1920.

5. Bought goods at 25, 20, 15, and 10% off. If the sum of my discounts amounted to $162.30, what was the list price of the goods? *Ans.* $300

IV. PROFIT AND LOSS.

1. ***Profit*** and ***Loss*** are terms which denote the gain or loss in business transactions.

2. ***Profit*** is the excess of the selling price above the cost.

3. ***Loss*** is the excess of the cost above the selling price.

I. A merchant reduced the price of a certain piece of cloth 5 cents per yard, and thereby reduced his profit on the cloth from 10% to 8%; what was the cost of the cloth per yard?

II.
1. 100%=cost of cloth per yard.
2. 10%=his profit before reduction.
3. 8%=his profit after reduction.
4. 10%—8%=2%=his reduction.
5. 5¢=reduction.
6. ∴ 2%=5¢,
7. 1%=$\frac{1}{2}$ of 5¢=$2\frac{1}{2}$¢, and
8. 100%=100 times $2\frac{1}{2}$¢=\$2.50=cost per yard.

III. ∴ \$2.50=cost of cloth per yard.

(*R. 3d p., p. 211, prob. 13.*)

I. A dealer sold two horses for \$150 each; on one he gained 25% and on the other he lost 25%; how much did he lose in the transaction?

II.
(1.) 100%=cost of the first horse;
(2.) 25%=gain.
(3.) 100%+25%=125%=selling price of first horse
(4.) \$150=selling price.
(5.) ∴ 125%=\$150,
(6.) 1%=$\frac{1}{125}$ of \$150=\$1.20, and
(7.) 100%=100 times \$1.20=\$120=cost of first horse.
(8.) \$150—\$120=\$30=gain on first horse.
(9.)
  1. 100%=cost of second horse.
  2. 25%=loss.
  3. 100%—25%=75%=selling price of second horse.
  4. \$150=selling price.
  5. ∴ 75%=\$150,
  6. 1%=$\frac{1}{75}$ of \$150=\$2, and
  7. 100%=100 times \$2=\$200=cost of second horse.

(10.) \$200—\$150=\$50=loss on second horse.
(11.) \$50—\$30=\$20=loss in the transaction.

III. ∴ He lost \$20 in the transaction.

(*R. 3d p., p. 211, prob. 12.*)

I. A speculator in real estate sold a house and lot for \$12000, which sale afford him a profit of $33\frac{1}{3}$% on the cost; he

then invested the \$12000 in city lots, which he was obliged to sell at a loss of $33\frac{1}{3}\%$; how much did he lose by the two transactions?

II.
- (1.) 100%=cost of the house and lot.
- (2.) $33\frac{1}{3}\%$=gain.
- (3.) $100\%+33\frac{1}{3}\%=133\frac{1}{3}\%$=selling price of house and lot.
- (4.) \$12000=selling price of the house and lot.
- (5.) $\therefore 133\frac{1}{3}\%$=\$12000.
- (6.) $1\%=\frac{1}{133\frac{1}{3}}$ of \$12000=\$90.
- (7.) 100%=100 times \$90=\$9000=cost of house and lot.
- (8.) \$12000—\$9000=\$3000=gain on house and lot.
- (9.)
  1. 100%=\$12000.
  2. $1\%=\frac{1}{100}$ of \$12000=\$120.
  3. $33\frac{1}{3}\%=33\frac{1}{3}$ times \$120=\$4000=loss on city lots.
- (10.) \$4000—\$3000=\$1000=loss by the two transactions.

III. $\therefore$ \$1000=his loss by the two transactions.

(*R. 3d p., p. 211, prob. 15.*)

I. A dealer sold two horses for the same price; on one he gained 20%, and on the other he lost 20%; his whole loss was \$25; what did each horse cost?

II.
- (1.) 100%=selling price of each horse.
- (2.)
  1. 100%=cost of first horse.
  2. 20%=gain on the first horse.
  3. $100\%+20\%=120\%$=selling price of first horse.
  4. $\therefore 120\%=100\%$, from (1),
  5. $1\%=\frac{1}{120}$ of $100\%=\frac{5}{6}\%$, and
  6. 100%=100 times $\frac{5}{6}\%=83\frac{1}{3}\%$=cost of first horse in terms of the selling price.
  7. $100\%-83\frac{1}{3}\%=16\frac{2}{3}\%$=gain on first horse.
- (3.)
  1. 100%=cost of the second horse.
  2. 20%=loss on second horse.
  3. $100\%-20\%=80\%$=selling price of second horse.
  4. $\therefore 80\%=100\%$, from (1),
  5. $1\%=\frac{1}{80}$ of $100\%=1\frac{1}{4}\%$, and
  6. 100%=100 times $1\frac{1}{4}\%=125\%$=cost of second horse in terms of the selling price.
  7. $125\%-100\%=25\%$=loss on the second horse.
- (4.) $25\%-16\frac{2}{3}\%=8\frac{1}{3}\%$=whole loss.
- (5.) \$25=whole loss.
- (6.) $\therefore 8\frac{1}{3}\%$=\$25,
- (7.) $1\%=\frac{1}{8\frac{1}{3}}$ of \$25=\$3, and
- (8.) 100%=100 times \$3=\$300=selling price of each horse.
- (9.) $83\frac{1}{3}\%=83\frac{1}{3}$ times \$3=\$250=cost of first horse.
- (10.) 125%=125 times \$3=\$375=cost of second horse.

III. ∴ $\begin{cases}\$250=\text{cost of the first horse, and}\\ \$375=\text{cost of second horse.}\end{cases}$

I. What % is lost if $\frac{2}{3}$ of cost equals $\frac{3}{4}$ of selling price?

II.
1. $\frac{3}{4}$ of selling price $=\frac{2}{3}$ of cost.
2. $\frac{1}{4}$ of selling price $=\frac{1}{3}$ of $\frac{2}{3}$ of cost $=\frac{2}{9}$ of cost.
3. $\frac{4}{4}$ of selling price $=4$ times $\frac{2}{9}$ of cost $=\frac{8}{9}$ of cost.
4. $\frac{9}{9}$=cost.
5. $\frac{8}{9}$=selling price.
6. $\frac{9}{9}-\frac{8}{9}=\frac{1}{9}$=loss.
7. $\frac{9}{9}=100\%$.
8. $\frac{1}{9}=\frac{1}{9}$ of $100\%=11\frac{1}{9}\%$, loss.

III. ∴ Loss $=11\frac{1}{9}\%$.

I. Paid \$125 for a horse, and traded him for another, giving 60% additional money. For the second horse I received a third and \$25. I then sold the third horse for \$150; what was my % of profit or loss?

II.
(1.) $100\%=\$125$,
(2.) $1\%=\frac{1}{100}$ of $\$125=\$1.25$, and
(3.) $60\%=60$ times $\$1.25=\$75$ = additional money paid for the second horse.
(4.) $\$125+\$75=\$200$=cost of second horse.
(5.) \$150=selling price of the third horse.
(6.) $\$150+\$25=\$175$=selling price of second horse.
(7.) $\$200-\$175=\$25$=loss in the transaction.
(8.)
 1. $\$200=100\%$,
 2. $\$1=\frac{1}{200}$ of $100\%=\frac{1}{2}\%$, ana
 3. $\$25=25$ times $\frac{1}{2}\%=12\frac{1}{2}\%$=my loss.

III. ∴ My loss is $12\frac{1}{2}\%$. *(R. H. A., p. 201, prob. 4.)*

I. If I buy at \$4 and sell at \$1, how many % do I lose?

II.
1. \$4=cost.
2. \$1=selling price.
3. $\$4-\$1=\$3$=loss.
4. $\$4=100\%$.
5. $\$1=\frac{1}{4}$ of $100\%=25\%$.
6. $\$3=3$ times $25\%=75\%$=loss.

III. ∴ 75%=loss.

I. A and B each lost \$5, which was $2\frac{7}{9}\%$ of A's and $3\frac{1}{3}\%$ of B's money; which had the most, and how much?

II.
- (1.) 100%=A's money.
- (2.) $2\frac{7}{9}\%$=what he lost.
- (3.) \$5=what he lost.
- (4.) $\therefore 2\frac{7}{9}\%=\$5$,
- (5.) $1\%=\frac{1}{2\frac{7}{9}}$ of \$5=\$1.80, and
- (6.) 100%=100 times \$1.80=\$180=A's money.
- (7.)
  1. 100%=B's money.
  2. $3\frac{1}{3}\%$=what he lost.
  3. \$5=what he lost.
  4. $\therefore 3\frac{1}{3}\%=\$5$,
  5. $1\%=\frac{1}{3\frac{1}{3}}$ of \$5=\$1.50, and
  6. 100%=100 times \$1.50=\$150=B's money.
- (8.) \$180—\$150=\$30=excess of A's money over B's.

III. ∴ A had \$30 more than B. (*R. H. A., p. 203, prob. 5.*)

I. Mr. A bought a horse and carriage, paying twice as much for the horse as for the carriage. He afterward sold the horse for 25% more than he gave for it, and the carriage for 20% less than he gave for it, receiving \$577.50; what was the cost of each?

II.
- (1.) 100%=cost of the carriage.
- (2.) 200%=cost of the horse.
- (3.) 20%=loss on the carriage.
- (4.) $100\%-20\%=80\%$=selling price of the carriage.
- (5.)
  1. $100\%=200\%$,
  2. $1\%=\frac{1}{100}$ of $200\%=2\%$, and
  3. 25%=25 times $2\%=50\%$=gain on the horse.
  4. $200\%+50\%=250\%$=selling price of the horse.
- (6.) $80\%+250\%=330\%$=selling price of both.
- (7.) \$577.50=selling price of both.
- (8.) $\therefore 330\%=\$577.50$,
- (9.) $1\%=\frac{1}{330}$ of \$577.50=\$1.75, and
- (10.) 100%=100 times \$1.75=\$175=cost of carriage.
- (11.) 200%=200 times \$1.75=\$350=cost of the horse.

III. ∴ \$175=cost of the carriage, and
\$350=cost of the horse.

(*Milne's prac., p. 259, prob. 19.*)

I. Mr. A. sold a horse for \$198, which was 10% less than he asked for him, and his asking price was 10% more than the horse cost him. What did the horse cost him?

II.
(1.) 100%=cost of the horse.
(2.) $100\%+10\%=110\%$=asking price.
(3.) 1. $100\%=110\%$,
2. $1\%=\frac{1}{100}$ of $110\%=1\frac{1}{10}\%$, and
3. $10\%=10$ times $1\frac{1}{10}\%=11\%$=reduction from asking price.
(4.) $110\%-11\%=99\%$=selling price.
(5.) \$198=selling price.
(6.) $\therefore 99\%=\$198$,
(7.) $1\%=\frac{1}{99}$ of \$198=\$2, and
(8.) $100\%=100$ times \$2=\$200=cost of the horse.

III. ∴ \$200=cost of horse. (*Milne's prac., p. 259, prob. 23.*)

I. What must be asked for apples which cost me \$3 per bbl., that I may reduce my asking price 20% and still gain 20% on the cost?

II.
(1.) $100\%=\$3$.
(2.) $1\%=\frac{1}{100}$ of \$3=\$.03, and
(3.) $20\%=20$ times \$.03=\$.60=gain.
(4.) \$3.00+\$.60=\$3.60=selling price.
(5.) 1. 100%=asking price.
2. 20%=reduction.
3. $100\%-20\%=80\%$=selling price.
4. \$3.60=selling price.
5. $\therefore 80\%=\$3.60$,
6. $1\%=\frac{1}{80}$ of \$3.60=\$.045, and
7. $100\%=100$ times \$.045=\$4.50=asking price.

III. ∴ \$4.50=asking price. (*Milne's prac., p. 261, prob. 38.*)

I. A merchant sold a quantity of goods at a gain of 20%. If, however, he had purchased them for \$60 less than he did, his gain would have been 25%. What did the goods cost him?

II.
(1.) 100%=actual cost of goods.
(2.) 20%=gain.
(3.) $100\%+20\%=120\%$=actual selling price.
(4.) $100\%-\$60$=supposed cost.
(5.) 1. $100\%=100\%-\$60$,
2. $1\%=\frac{1}{100}$ of $(100\%-\$60)=1\%-\$.60$, and
3. $25\%=25$ times $(1\%-\$.60)=25\%-\$15$=supposed gain.
(6.) $(100\%-\$60)+(25\%-\$15)=125\%-\$75$=actual selling price.
(7.) $\therefore 125\%-\$75=120\%$,
(8.) $5\%=\$75$,
(9.) $1\%=\frac{1}{5}$ of \$75=\$15, and
(10.) $100\%=100$ times \$15=\$1500=cost of the goods.

III. ∴ \$1500=cost of goods. (*Milne's prac., p. 261, prob. 40.*)

Note.—The selling price is the same in the last condition of this problem as in the first. Hence we have the selling price in the last condition equal to the selling price in the first as shown in step (7.)

I. I sold an article at 20% gain, had it cost me \$300 more, I would have lost 20%; find the cost.

II.
(1.) 100%=actual cost of the article.
(2.) 20%=actual gain.
(3.) 100%+20%=120%=actual selling price.
(4.) 100%+\$300=supposed cost.
(5.) 1. 100%=100%+\$300,
2. 1%=$\frac{1}{100}$ of (100%+\$300)=1%+\$3, and
3. 20%=20 times (1%+\$3)=20%+\$60=supposed loss.
(6.) (100%+\$300)—(20%+\$60)=80%+\$240=actual selling price.
(7.) ∴ 120%=80%+\$240.
(8.) 40%=\$240,
(9.) 1%=$\frac{1}{40}$ of \$240=\$6, and
(10.) 100%=100 times \$6=\$600=cost of the article.

III. ∴ \$600=cost of the article.

(*R. H. A., p. 409, prob. 85.*)

I. A man wishing to sell a horse and a cow, asked three times as much for the horse as for the cow, but, finding no purchaser, he reduced the price of the horse 20%, and the price of the cow 10%, and sold them for \$165. What did he get for each?

II.
(1.) 100%=asking price of the cow.
(2.) 300%=asking price of the horse.
(3.) 10%=reduction on the price of the cow.
(4.) 100%—10%=90%=selling price of the cow.
(5.) 1. 100%=300%,
2. 1%=$\frac{1}{100}$ of 300%=3%, and
3. 20%=20 times 3%=60%=reduction on horse.
(6.) 300%—60%=240%=selling price ot the horse.
(7.) 90%+240%=330%=selling price of both.
(8.) \$165=selling price of both
(9.) ∴ 330%=\$165,
(10.) 1%=$\frac{1}{330}$ of \$165=\$.50, and
(11.) 90%=90 times \$.50=\$45=selling price of cow.
(12.) 240%=240 times \$.50=\$120=selling price of horse.

III. ∴ \$45=amount he received for the cow, and
\$120=amount he received for the horse.

EXAMPLES.

1. What price must a man ask for a horse that cost him $200, that he may fall 20% on his asking price and still gain 20%? *Ans.* $300.

2. A man paid $150 for a horse which he offered in trade at a price he was willing to discount at 40% for cash, as he would then gain 20%. What was his trading price? *Ans.* $300.

3. A man gained 20% by selling his house for $3360. What did it cost him? *Ans.* $3000.

4. A gained 120% by selling sugar at 8¢ per pound. What did the sugar cost him per pound? *Ans.* $3\frac{7}{11}$¢.

5 How must cloth, costing $3.50 a yard, be marked that a merchant may deduct 15% from the marked price and still gain 15%? *Ans.* $\$4.73\frac{9}{11}$.

6. Sold a piece of carpeting for $240, and lost 20%; what selling price would have given me a gain of 20%? *Ans.* $360.

7. Sold two carriages for $240 apiece, and gained 20% on one and lost 20% on the other; how much did I gain or lose in the transaction? *Ans.* Lost $20.

8. Sold goods at a gain of 25% and investing the proceeds, sold at a loss of 25%; what was my % of gain or loss. *Ans.* $3\frac{1}{3}$%.

9. A man sold a horse and carriage for $597, gaining by the sale, 25% on the horse and 10% on the cost of the carriage. If $\frac{3}{4}$ of the cost of the horse equals $\frac{2}{3}$ of the the cost of carriage, what was the cost of each? *Ans.* Carriage $270; horse $240.

10. If $\frac{4}{9}$ of the selling price is gain, what is the profit? *Ans.* 80%.

11. If $\frac{1}{2}$ of an article be sold for the cost of $\frac{1}{3}$ of it, what is the rate of loss? *Ans.* $33\frac{1}{3}$%.

12. I sold two houses for the same sum; on one I gained 25% and on the other I lost 25%. My whole loss was $240; what did each house cost? *Ans.* First $1440, second $2400.

13. My tailor informs me that it will take $10\frac{1}{4}$ sq. yd. of cloth to make me a full suit of clothes. The cloth I am about to buy is $1\frac{7}{8}$ yards wide and on sponging it will shrink 5% in length and width. How many yards will it take for my new suit? *Ans.* $6\frac{62}{1083}$ yd.

14. A grocer buys coffee at 15¢ per ℔. to the amount of $90 worth, and sells it at the same price by Troy weight; find the % of gain or loss. *Ans.* Gain $21\frac{19}{36}$%.

15. I spent $260 for apples at $1.30 per bushel; after retaining a part for my own use, I sold the rest at a profit of 40%, clearing $3 on the whole cost. How many bushels did I buy?
Ans. 50 bu.

16 How must cloth costing $3.50 per yard, be marked that the merchant may deduct 15% from the marked price and still make 15% profit? *Ans.* $4.735.

17. I sold goods at a gain of 20%. If they had cost me $250 more than they did, I would have lost 20% by the sale· How much did the goods cost me? *Ans.* $500.

18. A merchant bought cloth at $3.25 per yard, and after keeping it 6 months sold it at $3.75 per yard. What was his gain %, reckoning 6% per annum for the use of money?
Ans. 12%+.

V. STOCKS AND BONDS.

1. ***Stocks*** is a general term applied to bonds, state and national, and to certificates of stocks belong to corporations.

3. ***A Bond*** is a written or printed obligation, under seal, securing the payment of a certain sum of money at or before a specified time.

3. ***Stock*** is the capital of the corporation invested in business; and is divided into *Shares*, usually of $100 each.

4. ***An Assessment*** is a sum of money required of the stockholders in proportion to their amount of stock.

5. ***A Dividend*** is a sum of money to be paid to the stockholders in proportion to their amounts of stock.

6. ***The Par Value*** of money, stocks, drafts, etc., is the nominal value on their face.

7. ***The Market Value*** is the sum for which they sell.

8. ***Discount*** is the excess of the par value of money, stocks, drafts, etc., over their market value.

9. ***Premium*** is the excess of their market value over their par value.

10. ***Brokerage*** is the sum paid an agent for buying stocks, bonds, etc.

I. At $\frac{1}{4}\%$ brokerage, a broker received \$10 for making an investment in bank stock; how many shares did he buy?

II.
1. 100%=par value of stock.
2. $\frac{1}{4}\%$=brokerage.
3. \$10=brokerage.
4. $\therefore$ $\frac{1}{4}\%$=\$10,
5. 1%=4 times \$10=\$40, and
6. 100%=100 times \$40=\$4000=par value of stock.
7. \$100=par value of one share.
8. \$4000=par value of 4000÷100, or 40 shares.

III. $\therefore$ 40=number of shares.

I. How many shares of railroad stock at 4% premium can be bought for \$9360?

II.
1. 100%=par value of stock I can buy.
2. 4%=premium.
3. 104%=price of what I buy.
4. \$9360=price of what I buy.
5. $\therefore$ 104%=\$9360.
6. 1%=$\frac{1}{104}$ of \$9360=\$90.
7. 100%=100 times \$90=\$9000=par value.
8. \$100=par value of one share.
9. \$9000=par value of 9000÷100, or 90 shares.

III. $\therefore$ 90=number of shares that can be bought.

I. When gold is at 105, what is the value of a gold dollar in currency?

II.
1. 105¢; or 105% in currency=100¢; or 100% in gold.
2. 1¢; or 1% in currency=.$95\frac{5}{21}$¢; or .$95\frac{5}{21}\%$ in gold.
3. 100¢; or 100% in currency=$95\frac{5}{21}$¢; or $95\frac{5}{21}\%$ in gold.

III. $\therefore$ \$1 in currency is worth $95\frac{5}{21}$¢ in gold.

I. In 1864, the "greenback" dollar was worth only $35\frac{5}{7}$¢ in gold; what was the price of gold?

II.
1. $35\frac{5}{7}$¢; or $35\frac{5}{7}\%$ in gold=100¢; or 100% in currency.
2. 1¢; or 1% in gold=$\frac{1}{35\frac{5}{7}}$ of 100¢; or 100%=2.8¢; or 2.8% in currency.
3. 100¢; or 100% in gold=100 times 2.8¢; or 2.8%=280¢; or 280% in currency.

III. $\therefore$ \$1 in gold was worth \$2.80 in currency.

(R. 3d p., p. 217, prob. 8.)

I. Bought stock at 10% discount, which rose to 5% premium and sold for cash. Paying a debt of \$33, I invested the balance in stock at 2% premium, which at par, left me \$11 less than at first; how much money had I at first?

(1.) 100%=my money at first.
(2.) 100%=par value of stock.
(3.) 10%=discount.
(4.) 100%—10%=90%=market value.
(5.) ∴ 90%=100%, my money; because that is the amount invested.
(6.) 1%=$\frac{1}{90}$ of 100%=$1\frac{1}{9}$%, and
(7.) 100%=100 times $1\frac{1}{9}$%=$111\frac{1}{9}$%=par value of the stock in terms of my money.
(8.) 1. 100%=$111\frac{1}{9}$%
2. 1%=$1\frac{1}{9}$%, and
3. 5%=5 times $1\frac{1}{9}$%=$5\frac{5}{9}$%=premium on stock in [terms of my money.
4. $111\frac{1}{9}$%+$5\frac{5}{9}$%=$116\frac{2}{3}$% = what I received for the stock.
5. $116\frac{2}{3}$%—$33=amount invested in second stock.

II. (9.) 1. 100%=par value of second stock.
2. 2%=premium.
3. 100%+2%=102%=market value of second stock.
4. $116\frac{2}{3}$%—$33=market value of second stock
5. ∴ 102%=$116\frac{2}{3}$%—$33,
6. 1%=$\frac{1}{102}$ of ($116\frac{2}{3}$%—$33)=$1\frac{22}{153}$%—$.$32\frac{6}{17}$,
7. 100%=100 times ($1\frac{22}{153}$%—$.$32\frac{6}{17}$)=$114\frac{58}{153}$%—$$32\frac{6}{17}$=par value of second stock.

(10.) $114\frac{58}{153}$%—$$32\frac{6}{17}$=what I received for the second stock, since I sold them at par.
(11.) ∴ $114\frac{58}{153}$%—$$32\frac{6}{17}$=100%—$11, by the last condition of the problem.
(12.) $14\frac{58}{153}$%=$$21\frac{6}{17}$,
(13.) 1%=$\dfrac{1}{14\frac{58}{153}}$ of $$21\frac{6}{17}$=$1.485, and
(14.) 100%=100 times $1.485=$148.50.

III. ∴ I had $148.50 at first. (*R. H. A., p. 212, prob. 8.*)

I. Bought $8000 in gold at 110%, brokerage $\frac{1}{8}$%; what did I pay for the gold in currency?

II. 1. 100%=par value of gold.
2. 110%=market value.
3. $\frac{1}{8}$%=brokerage.
4. 110%+$\frac{1}{8}$%=$110\frac{1}{8}$%=entire cost.
5. 100%=$8000,
6. 1%=$\frac{1}{100}$ of $8000=$80, and
7. $110\frac{1}{8}$%=$110\frac{1}{8}$ times $80=$8810=cost of gold in currency.

III. ∴ $8000 in gold costs $8810 in currency.

I. What income in currency would a man receive by investing $5220 in U. S. 5–20, 6% bonds at 116%, when gold is worth 105?

II.
- (1.) 100%=par value of the bonds.
- (2.) 116%=market value.
- (3.) \$5220=market value.
- (4.) ∴ 116%=\$5220.
- (5.) 1%=$\frac{1}{116}$ of \$5220=\$45.
- (6.) 100%=100 times \$45=\$4500=par value of bonds.
- (7.)
  1. 100%=\$4500.
  2. 1%=$\frac{1}{100}$ of \$4500=\$45.
  3. 6%=6 times \$45=\$270=income in gold.
- (8.) \$1.00 in gold=\$1.05 in currency.
- (9.) \$270 in gold=270 times \$1.05=\$283.50 in currency.

III. ∴ \$283.50=income in currency.

(*R. 3d p., p. 217, prob. 5.*)

I. What % of income do U. S. $4\frac{1}{2}$% bonds, at 108, yield when gold is 105%?

II.
- (1.) 100%=amount invested in the bonds.
- (2.) 100%=par value of bonds.
- (3.) 108%=market value.
- (4.) ∴ 108%=100%, from (1).
- (5.) 1%=$\frac{1}{108}$ of 100%=$\frac{25}{27}$%, [of amount invested.
- (6.) 100%=100 times $\frac{25}{27}$%=$92\frac{7}{9}$%=par value in terms
- (7.)
  1. 100%=$92\frac{7}{9}$%.
  2. 1%=$\frac{1}{100}$ of $92\frac{7}{9}$%=$\frac{25}{27}$%,
  3. $4\frac{1}{2}$%=$4\frac{1}{2}$ times $\frac{25}{27}$%=$4\frac{1}{6}$%=income in gold.
- (8.) 100% in gold=105% in currency.
- (9.) 1% in gold=$\frac{1}{100}$ of 105%=$1\frac{1}{20}$% in currency.
- (10.) $4\frac{1}{6}$% in gold=$4\frac{1}{6}$ times $1\frac{1}{20}$%=$4\frac{3}{8}$% in currency.

III. ∴ Income in currency=$4\frac{3}{8}$%.

Note.—This is a general solution of the preceding problem. Since there is no special amount given, we represent the amount invested by 100%. The market value and the amount invested being the same, we have 108%=100% as shown in (4).

I. A man bought Michigan Central at 120, and sold at 124%; what % of the investment did he gain?

II.
1. 124%=selling price.
2. 120%=cost.
3. 124%—120%=4%=gain.
4. 120%=100% of itself.
5. 1%=$\frac{1}{120}$ of 100%=$\frac{5}{6}$%,
6. 4%=4 times $\frac{5}{6}$%=$3\frac{1}{3}$%=gain on the investment.

III. ∴ He gained $3\frac{1}{3}$% on the investment.

I. What sum invested in U. S. 5's of 1881, at 118, yielded an annual income of \$1921 in currency, when gold was at 113?

II.
(1.) \$1.13 in currency=\$1 in gold.
(2.) \$1 in currency=$\frac{1}{1.13}$ of \$1=\$$\frac{100}{113}$ in gold, and
(3.) \$1921 in currency=1921 times \$$\frac{100}{113}$=\$1700=income in gold.
(4.) 100%=par value of the bonds.
(5.) 5%=income in gold.
(6.) \$1700=income in gold.
(7.) ∴ 5%=\$1700,
(8.) 1%=$\frac{1}{5}$ of \$1700=\$340, and
(9.) 100%=100 times \$340=\$34000=par value of the bonds.
(10.)
1. 100%=\$34000,
2. 1%=$\frac{1}{100}$ of \$34000=\$340, and
3. 118%=118 times \$340=\$40120=market value, or amount invested.

III. ∴ \$40120=amount invested.

SECOND SOLUTION.

II.
(1.) 100%=amount invested in currency.
(2.) 100%=par value.
(3.) 118%=market value.
(4.) ∴ 118%=100%, from (1.)
(5.) 1%=$\frac{1}{118}$ of 100%=$\frac{50}{59}$%, and
(6.) 100%=100 times $\frac{50}{59}$%=84$\frac{44}{59}$%=par value in terms of the investment.
(7.)
1. 100%=84$\frac{44}{59}$%.
2. 1%=$\frac{1}{100}$ of 84$\frac{44}{59}$%=$\frac{50}{59}$%, and
3. 5%=5 times $\frac{50}{59}$%=4$\frac{14}{59}$%=income in gold.
(8.)
1. 100% in gold=113% in currency,
2. 1% in gold=1$\frac{13}{100}$% in currency, and
3. 4$\frac{14}{59}$% in gold=4$\frac{14}{59}$ times 1$\frac{13}{100}$%=4$\frac{93}{118}$%=income in currency.
(9.) \$1921=income in currency.
(10.) ∴ 4$\frac{93}{118}$%=\$1921,
(11.) 1%=$\frac{1}{4\frac{93}{118}}$ of \$1921=\$401.20, and
(12.) 100%=100 times \$401.20=\$40120=amount invested in currency.

III. ∴ \$40120=amount invested. (*R. 3d p., p. 218, prob. 8.*)

I. How many shares of stock bought at 95$\frac{1}{4}$%, and sold at 105, brokerage $\frac{1}{4}$% on each transaction, will yield an income of \$925?

II.
1. 100%=par value of stock.
2. $95\frac{1}{4}\%$=market value of stock.
3. $\frac{1}{4}\%$=brokerage.
4. $95\frac{1}{4}+\frac{1}{4}\%=95\frac{1}{2}\%$=entire cost.
5. 105%=selling price+brokerage.
6. $\frac{1}{4}\%$=brokerage.
7. $105\%-\frac{1}{4}\%=104\frac{3}{4}\%$=selling price.
8. $104\frac{3}{4}\%-95\frac{1}{2}\%=9\frac{1}{4}\%$=gain.
9. $\$925$=gain.
10. $\therefore 9\frac{1}{4}\%=\$925$,
11. $1\%=\frac{1}{9\frac{1}{4}}$ of $\$925=\$100$, and
12. $100\%=100$ times $\$100=\$10000$=par value of stock.
13. $\$100$=par value one share.
14. $\$10000$=par value $10000\div100$, or 100 shares.

III. $\therefore$ 100=number of shares. (*R. 3d p., p. 218, prob. 9.*)

I. If I invest all my money in 5% furnace stock salable at 75%, my income will be \$180; how much must I borrow to make an investment in 5% state stock selling at 102%, to have that income?

II.
- (1.)
  1. 100%=par value of furnace stock.
  2. 5%=income.
  3. $\$180$=income.
  4. $\therefore 5\%=\$180$,
  5. $1\%=\frac{1}{5}$ of $\$180=\$36$, and [nace stock.
  6. $100\%=100$ times $\$36=\$3600$=par value of fur-
- (2.)
  1. $100\%=\$3600$,
  2. $1\%=\frac{1}{100}$ of $\$3600=\$36$, and [nace stock.

  \$. $75\%=75$ times $\$36=\$2700$=market value of fur-
- (3.)
  1. 100%=par value of state stock.
  2. 6%=income.
  3. $\$180$=income.
  4. $\therefore 6\%=\$180$,
  5. $1\%=\frac{1}{6}$ of $\$180=\$30$, and [stock.
  6. $100\%=100$ times $\$30=\$3000$=par value of state
- (4.)
  1. $100\%=\$3000$,
  2. $1\%=\frac{1}{100}$ of $\$3000=\$30$, and [state stock.
  3. $102\%=102$ times $\$30=\$3060$=market value of
- (5.) $\$3060-\$2700=\$360$=what I must borrow.

III. $\therefore$ I must borrow \$360. (*R. H. A., p. 225, prob. 2.*)

I. When U. S. 4% bonds are quoted at 106, what yearly income will be received in gold from bonds that can be bought for \$4982?

II. (1.) 100% = par value of the bonds.
(2.) 106% = market value.
(3.) \$4982 = market value, or amount invested.
(4.) ∴ 106% = \$4982,
(5.) 1% = $\frac{1}{106}$ of \$4982 = \$47, and
(6.) 100% = 100 times \$47 = \$4700.
(7.) 1. 100% = \$4700,
2. 1% = $\frac{1}{100}$ of \$4700 = \$47, and
3. 4% = 4 times \$47 = \$188 = income in gold.

III. ∴ \$188 = income in gold. (*R. 3p., p. 218, prob. 11.*)

I. The sale of my farm cost me \$500, but I gave the proceeds to a broker, allowing him $\frac{1}{2}$%, to purchase railroad stock then in the market at 102%; the farm paid 5% income, equal to \$2075, but the stock will pay \$2025 more; what is the rate of dividend?

II. (1.) 100% = value of the farm.
(2.) 5% = income on the farm.
(3.) \$2075 = income on the farm.
(4.) ∴ 5% = \$2075,
(5.) 1% = $\frac{1}{5}$ of \$2075 = \$415, and
(6.) 100% = 100 times \$415 = \$41500 = value of farm.
(7.) \$41500 — \$500 = \$41000 = amount invested in stock.
(8.) 1. 100% = par value of the stock.
2. 102% = market value, or amount invested.
3. $\frac{1}{2}$% = brokerage
4. 102% + $\frac{1}{2}$% = 102$\frac{1}{2}$% = entire cost of stock.
5. ∴ 102$\frac{1}{2}$% = \$41000,
6. 1% = $\frac{1}{102\frac{1}{2}}$ of \$41000 = \$400, and
7. 100% = 100 times \$400 = \$40000 = par value of the railroad stock.
(9.) 1. \$2075 + \$2025 = \$4100 = income on railroad stock.
2. \$40000 = 100%,
3. \$1% = $\frac{1}{40000}$ of 100% = $\frac{1}{400}$%, and
4. \$4100 = 4100 times $\frac{1}{400}$% = 10$\frac{1}{4}$% = rate of dividend.

III. ∴ 10$\frac{1}{4}$% = rate of dividend. (*R. H. A., p. 224, prob. 4.*)

I. What must be paid for 6% bonds to realize an income of 8% on the investment?

II. 1. 100% = amount invested.
2. 6% = income on the par value of the bonds.
3. 8% = income on the investment.
4. ∴ 8% of investment = 6% of the par value,
5. 1% of investment = $\frac{1}{8}$ of 6% = $\frac{3}{4}$% of the par value, and
6. 100% of investment = 100 times $\frac{3}{4}$% = 75% of par value.

III. ∴ Must pay 75% to make 8% on the investment.

Note.—It must be borne in mind that 100% of any quantity is the quantity itself. ∴ 100% of the amount invested equals the

amount invested. It must also be remembered that the income on the par value is equal to the income on the investment. Suppose I buy a 500-dollar 6% bond for \$400. The income on the par value, or face of the bond is 6% of \$500, or \$30. But \$30 is $7\frac{1}{2}$% of \$400, the amount invested. Hence, the truth of step 4 in the above solution.

I. Which is the better investment, buying 9% stock at 25% advance, or 6% stock at 25% discount.

II. A.
(1.) 100%=amount invested in the 9% stock,
(2.) 100%=par value.
(3.) 25%=premium.
(4.) 100%+25%=125%=market value.
(5.) ∴ 125%=100%,
(6.) 1%=$\frac{1}{125}$ of 100%=$\frac{4}{5}$%, and
(7.) 100%=100 times $\frac{4}{5}$%=80%=par value in terms of the investment.
(8.) 1. 100%=80%,
2. 1%=$\frac{1}{100}$ of 80%=$\frac{4}{5}$%, and
3. 9%=9 times $\frac{4}{5}$%=$7\frac{1}{5}$%=income of 9% stock.

B.
(1.) 100%=amount invested in 6% stock.
(2.) 100%=par value of 6% stock.
(3.) 25%=discount.
(4.) 100%—25%=75%=market value.
(5.) ∴ 75%=100%.
(6.) 1%=$\frac{1}{75}$ of 100%=$1\frac{1}{3}$%, and
(7.) 100%=100 times $1\frac{1}{3}$%=$133\frac{1}{3}$%=par value of the 6% stock in terms of the investment.
(8.) 1. 100%=$133\frac{1}{3}$%,
2. 1%=$\frac{1}{100}$ of $133\frac{1}{3}$%=$1\frac{1}{3}$%, and
3. 6%=6 times $1\frac{1}{3}$%=8%=income of 6% stock.

III. ∴ The latter is the better investment, since it pays 8%—$7\frac{1}{5}$%, or $\frac{4}{5}$% more income on the investment.

(*Greenleaf's N. A., p. 298, prob. 5.*)

I. If I pay $87\frac{1}{2}$% for railroad bonds that yield an annual income of 7%, what % do I get on my investment?

II.
(1.) 100%=investment.
(2.) 100%=par value.
(3.) $87\frac{1}{2}$%=market value, or amount invested.
(4.) ∴ $87\frac{1}{2}$%=100%, from (1.)
(5.) 1%=$\frac{1}{87\frac{1}{2}}$ of 100%=$1\frac{1}{7}$%, and
(6.) 100%=100 times $1\frac{1}{7}$%=$114\frac{2}{7}$%=par value in terms of the investment.
(7.) 1. 100%=$114\frac{2}{7}$%,
2. 1%=$\frac{1}{100}$ of $114\frac{2}{7}$%=$1\frac{1}{7}$%, and
3. 7%=7 times $1\frac{1}{7}$%=8%=income on the investment.

III. ∴ 8%=income on the investment.

I. A banker owns $2\frac{1}{2}\%$ stocks at 10% below par, and 3% stocks at 15% below par. The income from the former is $66\frac{2}{3}\%$ more than from the latter, and the investment in the latter is \$11400 less than in the former; required the whole investment and income.

II.

(1.) 100%=investment in the former.

(2.) $100\%-\$11400$=investment in the latter.

(3.)
1. 100%=par value of the former.
2. 10%=discount of the former.
3. $100\%-10\%=90\%$=market value, or amount in- [vested in former.
4. $\therefore 90\%=100\%$, from (1),
5. $1\%=\frac{1}{90}$ of $100\%=1\frac{1}{9}\%$, and
6. $100\%=100$ times $1\frac{1}{9}\%=111\frac{1}{9}\%$=par value of former in terms of the investment.

(4.)
1. $100\%=111\frac{1}{9}\%$,
2. $1\%=\frac{1}{100}$ of $111\frac{1}{9}\%=1\frac{1}{9}\%$, and
3. $2\frac{1}{2}\%=2\frac{1}{2}$ times $1\frac{1}{9}\%=2\frac{7}{9}\%$=income of former in terms of the investment.

(5.)
1. 100%=par value of the latter.
2. 15%=discount.
3. $100\%-15\%=85\%$=market value, or amount in- [vested in the latter.
4. $\therefore 85\%=100\%-\$11400$, from (2),
5. $1\%=\frac{1}{85}$ of $(100\%-\$11400)=1\frac{3}{17}\%-\$134\frac{2}{17}$,
6. $100\%=100$ times $(1\frac{3}{17}\%-\$134\frac{2}{17})=117\frac{11}{17}\%-\$13411\frac{13}{17}$=par value of latter in terms of former.

(6.)
1. $100\%=117\frac{11}{17}\%-\$13411\frac{13}{17}$,
2. $1\%=\frac{1}{100}$ of $(117\frac{11}{17}\%-\$13411\frac{13}{17})=1\frac{3}{17}\%-\$134\frac{2}{17}$, and
3. $3\%=3$ times $(1\frac{3}{17}\%-\$134\frac{2}{17})=3\frac{9}{17}\%-\$402\frac{6}{17}$ =income of latter in terms of the investment.

(7.)
1. 100%=income of the latter.
2. $100\%+66\frac{2}{3}\%=166\frac{2}{3}\%$=income of the former.
3. $2\frac{7}{9}\%$=income of the former.
4. $\therefore 166\frac{2}{3}\%=2\frac{7}{9}\%$,
5. $1\%=\dfrac{1}{166\frac{2}{3}}$ of $2\frac{7}{9}\%=\frac{1}{60}\%$, and
6. $100\%=100$ times $\frac{1}{60}\%=1\frac{2}{3}\%$=income of latter in [terms of income of former.

(8.) $3\frac{9}{17}\%-\$402\frac{6}{17}$=income of the latter.

(9.) $\therefore 3\frac{9}{17}\%-\$402\frac{6}{17}=1\frac{2}{3}\%$.

(10.) $1\frac{44}{51}\%=\$402\frac{6}{17}\%$,

(11.) $1\%=\dfrac{1}{1\frac{44}{51}}$ of $\$402\frac{6}{17}=\$216$,

(12.) $100\%=100$ times $\$216=\$21600$=investment in [former.

(13.) $100\%-\$11400=\$21600-\$11400=\$10200$=investment in latter.

(14.) $2\frac{7}{9}\%=2\frac{7}{9}$ times $\$216=\$600$=income of former.

(15.) $3\frac{9}{17}\%-\$402\frac{6}{17}=3\frac{9}{17}$ times $\$216-\$402\frac{6}{17}=\$360=$ income of latter.

(16.) $\$21600+\$10200=\$31800=$whole investment.

(17.) $\$600+\$360=\$960=$whole income.

III. $\therefore\begin{cases}\$31800=\text{whole investment, and}\\ \$960=\text{whole income.}\end{cases}$ (*R. H. A., p. 225, prob. 4.*)

I. W. F. Baird, through his broker, invested a certain sum of money in Philadelphia 6's at $115\frac{1}{2}\%$, and three times as much in Union Pacific 7's at $89\frac{1}{2}\%$, brokerage $\frac{1}{2}\%$ in both cases; how much was invested in each kind of stock if his annual income is \$9920?

II.

(1.) $100\%=$amount invested in Philadelphia 6's.

(2.) $300\%=$amount invested in Union Pacific 7's.

(3.)
1. $100\%=$par value of Philadelphia 6's.
2. $115\frac{1}{2}\%=$market value.
3. $\frac{1}{2}\%=$brokerage.
4. $115\frac{1}{2}\%+\frac{1}{2}\%=116\%=$entire cost of Phila. 6's.
5. $\therefore\ 116\%=100\%$.
6. $1\%=\frac{1}{116}$ of $100\%=\frac{25}{29}\%$, and
7. $100\%=100$ times $\frac{25}{29}\%=86\frac{6}{29}\%=$par value of Philadelphia 6's in terms of investment.

(4.)
1. $100\%=86\frac{6}{29}\%$,
2. $1\%=\frac{1}{100}$ of $86\frac{6}{29}\%$, and
3. $6\%=6$ times $\frac{25}{29}\%=5\frac{5}{29}\%=$income of Philadelphia 6's in terms of investment.

(5.)
1. $100\%=$par value of Union Pacific 7's.
2. $89\frac{1}{2}\%=$market value.
3. $\frac{1}{2}\%=$brokerage.
4. $89\frac{1}{2}\%+\frac{1}{2}\%=90\%=$entire cost of Union Pacific 7's.
5. $\therefore\ 90\%=300\%$,
6. $1\%=\frac{1}{90}$ of $300\%=3\frac{1}{3}\%$, and
7. $100\%=100$ times $3\frac{1}{3}\%=333\frac{1}{3}\%=$par value of Union Pacific 7's.

(6.)
1. $100\%=333\frac{1}{3}\%$,
2. $1\%=\frac{1}{100}$ of $333\frac{1}{3}\%=3\frac{1}{3}\%$, and
3. $7\%=7$ times $3\frac{1}{3}\%=23\frac{1}{3}\%=$income of Union Pacific 7's in terms of investment.

(7.) $5\frac{5}{29}\%+23\frac{1}{3}\%=28\frac{44}{87}\%=$whole income.

(8.) $\$9920=$whole income.

(9.) $\therefore\ 28\frac{44}{87}\%=\$9920$,

(10.) $1\%=\dfrac{1}{28\frac{44}{87}}$ of $\$9920=\$348$,

(11.) $100\%=100$ times $\$348=\$34800=$amount invested [in Philadelphia 7's.

(12.) $300\%=300$ times $\$348=\$104500=$ amount invested in Union Pacific 7's.

III. $\therefore$ $\begin{cases}\$34800=\text{amount invested in Philadelphia 6's, and}\\ \$104400=\text{amount invested in Union Pacific 7's.}\end{cases}$

(*R. H. A., p. 225, prob. 6.*)

I. Thomas Reed bought 6% mining stock at $114\frac{1}{2}\%$, and 4% furnace stock at 112%, brokerage $\frac{1}{2}\%$; the latter cost him \$430 more than the former, but yielded the same income; what did each cost him?

II.

(1.) $100\%=$ amount invested in mining stock.

(2.) $100\%+\$430=$ amount invested in furnace stock.

(3.)
1. $100\%=$ par value of mining stock.
2. $114\frac{1}{2}\%=$ market value.
3. $\frac{1}{2}\%=$ brokerage.
4. $114\frac{1}{2}\%+\frac{1}{2}\%=115\%=$ entire cost.
5. $\therefore$ $115\%=100\%$, from (1),
6. $1\%=\frac{1}{115}$ of $100\%=\frac{20}{23}\%$, and
7. $100\%=100$ times $\frac{20}{23}\%=96\frac{22}{23}\%=$ par value of mining stock in terms of investment.

(4.)
1. $100\%=96\frac{22}{23}\%$,
2. $1\%=\frac{1}{100}$ of $96\frac{22}{23}\%=\frac{20}{23}\%$, and
3. $6\%=6$ times $\frac{20}{23}\%=5\frac{5}{23}\%=$ income of mining stock in terms of investment.

(5.)
1. $100\%=$ par value of furnace stock.
2. $112\%=$ market value.
3. $\frac{1}{2}\%=$ brokerage.
4. $112\%+\frac{1}{2}\%=112\frac{1}{2}\%=$ entire cost.
5. $\therefore$ $112\frac{1}{2}\%=100\%+\$430$,
6. $1\%=\dfrac{1}{112\frac{1}{2}}$ of $(100\%+\$430)=\frac{8}{9}\%+\$3\frac{37}{45}$, and
7. $100\%=100$ times $(\frac{8}{9}\%+\$3\frac{37}{45})=88\frac{8}{9}\%+\$382\frac{2}{9}=$ par value of furnace stock in terms of investm't.

(6.)
1. $100\%=88\frac{8}{9}\%+\$382\frac{2}{9}$,
2. $1\%=\frac{1}{100}$ of $(88\frac{8}{9}\%+\$382\frac{2}{9})=\frac{8}{9}\%+\$3\frac{37}{45}$, and
3. $4\%=4$ times $(\frac{8}{9}\%+\$3\frac{37}{45})=3\frac{5}{9}\%+\$15\frac{13}{45}=$ income of furnace stock in terms of the investment.

(7.) $\therefore$ $5\frac{5}{23}\%=3\frac{5}{9}\%+\$15\frac{13}{45}$, by the conditions of the problem,

(8.) $1\frac{137}{207}\%=\$15\frac{13}{45}$,

(9.) $1\%=\dfrac{1}{1\frac{137}{207}}$ of $\$15\frac{13}{45}=\$9.20$, and

(10.) $100\%=100$ times $\$9.20=\$920=$ amount invested in mining stock.

(11.) $100\%+\$430=\$1350=$ amount invested in furnace stock.

(*R. H. A., p. 225, prob. 7.*)

III. $\therefore$ $\begin{cases}\$920=\text{amount invested in mining stock, and}\\ \$1350=\text{amount invested in furnace stock.}\end{cases}$

I. Suppose 10% state stock is 20% better in market than 4% railroad stock; if A.'s income be $500 from each, how much money has he paid for each, the whole investment bringing $6\frac{2}{333}\%$?

II.

(1.)
1. 100%=par value of state stock.
2. 10%=income.
3. $500=income.
4. ∴ 10%=$500,
5. 1%=$\frac{1}{10}$ of $500=$50, and
6. 100%=100 times $50=$5000=par value of state stock.

(2.)
1. 100%=par value of railroad stock.
2. 4%=income.
3. $500=income.
4. ∴ 4%=$500,
5. 1%=$\frac{1}{4}$ of $500=$125, and
6. 100%=100 times $125=$12500=par value of railroad stock.

(3.) $5000=$\frac{2}{5}$ of $12500, *i. e.*, the face of state stock is $\frac{2}{5}$ of face of railroad stock.

(4.)
1. 100%=whole investment.
2. $6\frac{2}{333}\%$=income of whole investment.
3. $500+$500=$1000=income of whole investment.
4. ∴ $6\frac{2}{333}\%$=$1000,
5. 1%=$\frac{1}{6\frac{2}{333}}$ of $1000=$166.50, and
6. 100%=100 times $166.50=$16650=whole investment.

(5.)
1. 100%=investment in railroad stock.
2.
   1′. 40%=$\frac{2}{5}$ of 100%=investment in state stock, excluding the 20% excess.
   2′. 100%=40%,
   3′. 1%=$\frac{1}{100}$ of 40%=$\frac{2}{5}$%, and
   4′. 20%=20 times $\frac{2}{5}$% =8%=excess of state stock over same amount of railroad stock.
3. 40%+8%=48%=investment in state stock.
4. 100%+48%=148%=whole investment.
5. $16650=whole investment.
6. ∴ 148%=$16650,
7. 1%=$\frac{1}{148}$ of $16650=$112.50,
8. 100%=100 times $112.50=$11250=investment in railroad stock.
9. 48%=48 times $112.50=$5400=investment in state stock

III. ∴ $11250=amount invested in railroad stock, and
$5400=amount invested in state stock.

(*R. H. A., p. 227, prob. 5.*)

EXAMPLES.

1. What could I afford to pay for bonds yielding an annual income of 9% to invest my money so as to realize 6% on the investment? *Ans.* 150%.

2. What must I pay for Chicago, Burlington & Quincy Railroad stock that bears 6% that my annual income on the investment may yield 5%? *Ans.* 120%.

3. Bought 75 shares N. Y., P. & O. Railroad stock at 105%, and sold them at $108\frac{1}{2}$% ; how much did I gain in the transaction? *Ans.* $262.50.

4. How many shares of bank stock at 5% premium, can be bought for $7665? *Ans.* 73.

5. A broker bought stock at 4% discount, and, selling them at 3% premium, gained $1400 ; how many shares did he buy? *Ans.* 200.

6. At what price must I buy 15% stock that it may yield the same income as 4% stock purchased at 90%? *Ans.* $337\frac{1}{2}$%.

7. How much must I pay for New York 6's so that I may realize an income of 90%. *Ans.* $66\frac{2}{3}$%.

8. At what price must I buy 7% stock so that they may yield an income equivalent to 10% stocks at par? *Ans.* 70%.

9. What sum must I invest in U. S. 6's at 112% to secure an annual income of $1800? *Ans.* $35400.

10. Which is the more profitable, and how much, to invest $5000 in 6% stock purchased at 75%, or 5% stock purchased at 60%? *Ans*, The latter; $$16\frac{2}{3}$.

11. If a man who had $5000 U. S. 6's of 1881 should sell them at 115%, and invest in U. S. 10-40's purchased at 105%, would he gain or lose and how much? *Ans.* Loss $26.19.

12. When gold is at 120, what is a "greenback" dollar worth? *Ans.* $83\frac{1}{3}$¢.

13. Suppose the market value of 5% bank stock to be $11\frac{1}{9}$% higher than 8% corporation bonds ; I realize 8% on my investment, and my income from each is $180; what did I invest in each? *Ans.* $$2923.07\frac{9}{13}$ in former, and $$1576.92\frac{4}{13}$ in latter.

14. A bought 5% railroad stock at $109\frac{1}{2}$%, and $4\frac{1}{2}$% pike stock at $107\frac{1}{2}$%, brokerage $\frac{1}{2}$%; the former cost $100 less than the latter but yielded the same income; what did each cost him? *Ans.* $1100 cost of former, and $1200 cost of latter.

15. What rate % of income shall I receive if I buy U. S. 5's at a premium of 10%, and receive payment at par in 15 years?
Ans. $3\frac{31}{33}\%$.

16. Suppose the market value of 6% corporation stock is 20% less than 5% state stock; if my income be \$1200 from each, what did I pay for each if the whole investment brings 6%?
Ans. \$16000, and \$24000.

17. I bought $2\frac{1}{2}\%$ stock at 80%, and $4\frac{1}{2}\%$ stock at 86%. The income on the former was $44\frac{4}{9}\%$ more than on the latter, but my investment is \$22140 less in the latter than in the former; what do I realize on my investment? *Ans.* $3\frac{221}{293}\%$.

Hint.—Find the whole investment, and whole income as in the problem on page 75. Then find what % the whole income is of the whole investment.

18. Invested in U. S. $4\frac{1}{2}$'s at 105, brokerage $\frac{1}{2}\%$; $\frac{4}{5}$ as much in U. P. 6's at $119\frac{2}{3}$, brokerage $\frac{1}{3}\%$; and 3 times as much in N. Y. 7's, brokerage $\frac{1}{4}\%$. If my entire income is \$1702, find my investment. *Ans.* \$25320.

19. A. paid \$1075 for U. S. 5-20 6% bonds at $7\frac{1}{2}\%$ premium, interest payable semi-annually in gold. When the average premium on gold was 112%, did he make more or less than B. who invested an equal sum in railroad stock at 14% below par, which paid a semi-annual dividend of 4%?
Ans. A. makes \$16.40 less than B. every six months.

20. I invested \$4200 in railroad stock at 105, and sold it at 80%; how much must I borrow at 4% so that by investing all I have in 6% bonds at 7% interest, payable annually, I may retrieve my loss in one year? *Ans.* \$18600.

VII. INSURANCE.

1. ***Insurance*** is indemnity against loss or damage.

2. Insurance.
- 1. Property Insurance.
  - 1. Fire Insurance.
  - 2. Marine Insurance.
- 2. Personal Insurance.
  - 1. Life Insurance.
  - 2. Accident Insurance.
  - 3. Health Insurance.

3. ***Property Insurance*** is the indemnity against loss or damage of property.

4. ***Personal Insurance*** is indemnity against loss of life or health.

5. ***Fire Insurance*** is indemnity against loss by fire.

6. ***Marine Insurance*** is indemnity against the dangers of navigation

7. ***Life Insurance*** is a contract in which a company agrees, in consideration of certain premiums received, to pay a certain sum to the heirs or assigns of the insured at his death, or to himself if he attains a certain age.

8. ***Accident Insurance*** is indemnity against loss by accident.

9. ***Health Insurance*** is a weekly indemnity in case of sickness.

10. ***The Insurer***, or ***Underwriter***, is the party, or company, that undertakes the risk.

11. ***The Risk*** is the particular danger against which the insurer undertakes.

12. ***The Insured*** is the party protected against loss.

13. ***The Premium*** is the sum paid for insurance; and is a certain per cent. of the amount insured.

14. ***The Amount***, or ***Valuation***, is the sum for which the premium is paid.

I. My house is permanently insured for \$1800, by a deposit of ten annual premiums, the rate per year being $\frac{3}{4}\%$; how much did I deposit, and if, on terminating the insurance, I receive my deposit less 5%; how much do I get?

II.
(1.) $100\%=\$1800$,
(2.) $1\%=\frac{1}{100}$ of $\$1800=\$18$, and
(3.) $\frac{3}{4}\%=\frac{3}{4}$ times $\$18=\$13.50$=one annual deposit.
(4.) $\$135=10$ times $\$13.50$=ten annual deposits.
(5.)
1. $100\%=\$135$,
2. $1\%=\frac{1}{100}$ of $\$135=\$1.35$, and
3. $5\%=5$ times $\$1.35=\$6.75$=deduction.
(6.) $\$135-\$6.75=\$128.25$=what I received.

III. ∴ \$135=amount deposited, and
\$128.25=amount received.

(*R. H. A., p. 230, prob. 5.*)

I. An insurance company having a risk of \$25000, at $\frac{9}{10}\%$, reinsured \$10000, at $\frac{4}{5}\%$, with another office, and \$5000, at 1%, with another; how much did it clear above what it paid?

II.
(1.) $100\%=\$25000$,
(2.) $1\%=\frac{1}{100}$ of $\$25000=\$250$, and
(3.) $\frac{9}{10}\%=\frac{9}{10}$ times $\$250=\$225$=what the company received for taking the risk.
(4.)
1. $\$10000$=amount the company reinsured at $\frac{4}{5}\%$.
2. $100\%=\$10000$,
3. $1\%=\frac{1}{100}$ of $\$10000=\$100$, and
4. $\frac{4}{5}\%=\frac{4}{5}$ times $\$100=\$80$=what the company paid for reinsuring $10000.
(5.)
1. $\$5000$=amount reinsured in another office at 1%.
2. $100\%=\$5000$,
3. $1\%=\frac{1}{100}$ of $\$5000=\$50$=what the company paid for reinsuring $5000.
4. $\$80+\$50=\$130$=what the company paid out.
5. $\$225-\$130=\$95$=what it cleared.

III. $\therefore$ $95=what the company cleared.

(*R. H. A., p. 230, prob. 7.*)

I. I took a risk at $4\frac{1}{2}\%$; reinsured $\frac{2}{5}$ of it at 2%, and $\frac{1}{4}$ of it at $2\frac{1}{2}\%$; what rate of insurance do I get on what is left?

II.
(1.) 100%=whole risk.
(2.) $1\frac{1}{2}\%$=premium.
(3.)
1. $40\%=\frac{2}{5}$ of 100%=amount reinsured at 2%.
2. $100\%=40\%$,
3. $1\%=\frac{1}{100}$ of $40\%=\frac{2}{5}\%$, and
4. $2\%=2$ times $\frac{2}{5}\%=\frac{4}{5}\%$=amount I pay out for reinsuring $\frac{2}{5}$ of the risk.
(4.)
1. $25\%=\frac{1}{4}$ of 100%=second part reinsured.
2. $100\%=25\%$.
3. $1\%=\frac{1}{100}$ of $25\%=\frac{1}{4}\%$, and
4. $2\frac{1}{2}\%=2\frac{1}{2}$ times $\frac{1}{4}\%=\frac{5}{8}\%$=amount I paid out for reinsuring $\frac{1}{4}$ of the risk.
(5.) $\frac{4}{5}\%+\frac{5}{8}\%=1\frac{17}{40}\%$=amount of premiums paid out.
(6.) $1\frac{1}{2}\%-1\frac{17}{40}\%=\frac{3}{40}\%$=amount of premium I had left.
(7.) $40\%+25\%=65\%$=whole amount reinsured.
(8.) $100\%-65\%=35\%$=risk left on which I received $\frac{3}{40}\%$ premium.
(9.)
1. $35\%=100\%$ of itself.
2. $1\%=\frac{1}{35}$ of $100\%=2\frac{6}{7}\%$, and
3. $\frac{3}{40}\%=\frac{3}{40}$ times $2\frac{6}{7}\%=\frac{3}{14}\%$=rate of premium

III. $\therefore$ $\frac{3}{14}\%$=rate of insurance I receive.

(*R. H. A., p. 231, prob. 6.*)

Remark.—35% is the base and $\frac{3}{40}\%$ is the percentage, and we wish to know what per cent. $\frac{3}{40}\%$ is of 35%.

I. Took a risk at 2%; reinsured $10000 of it at $2\frac{1}{8}\%$ and $8000 at $1\frac{3}{4}\%$; my share of the premium was $207.50; what sum was insured?

II.
- (1.)
  1. 100%=\$10000,
  2. 1%=$\frac{1}{100}$ of \$10000=\$100, and [\$10000 reinsured.
  3. $2\frac{1}{8}$%=$2\frac{1}{8}$ times \$100=\$212.50=amount paid out on
- (2.)
  1. 100%=\$8000,
  2. 1%=$\frac{1}{100}$ of \$8000=\$80, and [\$8000 reinsured.
  3. $1\frac{3}{4}$%=$1\frac{3}{4}$ times \$80=\$140=amount paid out on
- (3.) \$212.50+\$140=\$352.50=whole amount paid out.
- (4.) \$207.50=what I realize.
- (5.) ∴ \$352.50+207.50=\$560=premium on whole risk.
- (6.) 100%=risk.
- (7.) 2%=premium.
- (8.) \$560=premium.
- (9.) ∴ 2%=\$560,
- (10.) 1%=$\frac{1}{2}$ of \$560=\$280, and
- (11.) 100%=100 times \$280=\$28000=risk.

III. ∴ \$28000=risk. (*R. H. A., p. 232, prob. 6.*)

I. I can insure my house for \$2500 at $\frac{8}{10}$% premium annually, or permanently by paying down 12 annual premiums; which should I prefer, and how much will I gain by it if money is worth 6% per annum to me?

II
- (1.) 100%=\$2500.
- (2.) 1%=$\frac{1}{100}$ of \$2500=\$25, and
- (3.) $\frac{8}{10}$%=$\frac{8}{10}$ times \$25=\$20=one annual premium.
- (4.) \$240=12 times \$20=twelve annual premiums.
- (5.)
  1. 100%=the amount that will produce \$20 annually at 6%.
  2. 6%=interest.
  3. \$20=interest.
  4. ∴ 6%=\$20,
  5. 1%=$\frac{1}{6}$ of \$20=\$$3\frac{1}{3}$, and
  6. 100%=100 times \$$3\frac{1}{3}$=\$$333\frac{1}{3}$=the amount that will produce \$20 annually at 6%.
- (6.) \$$333\frac{1}{3}$+\$20=\$$353\frac{1}{3}$=amount I would have to pay down by the former condition. [tion.
- (7.) ∴ \$$353\frac{1}{3}$—\$240=\$$113\frac{1}{3}$=gain by the latter condi-

III. ∴ The latter is the better.
\$$113\frac{1}{3}$=gain.

Remark.—In (6) we add \$20, since a payment must be made immediately. \$$333\frac{1}{3}$ will not produce that sum until the end of the year.

I. The Mutual Fire Insurance Company insured a building and its stock for $\frac{2}{3}$ of its value, charging $1\frac{3}{4}\%$. The Union Insurance Company relieved them of $\frac{1}{4}$ of the risk, at $1\frac{1}{2}\%$. The building and stock being destroyed by fire, the Union lost \$49000 less than the Mutual; what amount of money did the owners of the building and stock lose?

II.
(1.) 100%=value of the building and stock.
(2.) $66\frac{2}{3}\%=\frac{2}{3}$ of 100%=amount insured.
(3.) $1\frac{3}{4}\%$=rate of insurance.
(4.) 1. $100\%=66\frac{2}{3}\%$,
2. $1\%=\frac{1}{100}$ of $66\frac{2}{3}\%=\frac{2}{3}\%$, and
3. $1\frac{3}{4}\%=1\frac{3}{4}$ times $\frac{2}{3}\%=1\frac{1}{6}\%$=what Mutual received from the owners of the building and stock.
(5.) $16\frac{2}{3}\%=\frac{1}{4}$ of $66\frac{2}{3}\%$=amount of which the Union relieved the Mutual.
(6.) 1. $100\%=16\frac{2}{3}\%$,
2. $1\%=\frac{1}{100}$ of $16\frac{2}{3}\%=\frac{1}{6}\%$, and
3. $1\frac{1}{2}\%=1\frac{1}{2}$ times $\frac{1}{6}\%=\frac{1}{4}\%$=what the Mutual paid the Union for taking the risk of $16\frac{2}{3}\%$.
(7.) $16\frac{2}{3}\%+1\frac{1}{6}\%=17\frac{5}{6}\%$=whole amount the Mutual received.
(8.) $66\frac{2}{3}\%+\frac{1}{4}\%=66\frac{11}{12}\%$=whole amount the Mutual paid out.
(9.) $\therefore 66\frac{11}{12}\%-17\frac{5}{6}\%=49\frac{1}{12}\%$=amount the Mutual lost.
(10.) $16\frac{2}{3}\%$=amount the Union paid the Mutual.
(11.) $\frac{1}{4}\%$=amount the Union received from the Mutual.
(12.) $\therefore 16\frac{2}{3}\%-\frac{1}{4}\%=16\frac{5}{12}\%$=amount the Union lost.
(13.) $49\frac{1}{12}\%-16\frac{5}{12}\%=32\frac{2}{3}\%$=what the Mutual lost more than the Union.
(14.) \$49000=what the Mutual lost more than the Union.
(15.) $\therefore 32\frac{2}{3}\%=\$49000$,
(16.) $1\%=\frac{1}{32\frac{2}{3}}$ of \$49000=\$1500, and
(17.) 100%=100 times \$1500=\$150000=value of building and stock.
(18.) $66\frac{2}{3}\%=66\frac{2}{3}$ times \$1500=\$100000=amount insured.
(19.) $33\frac{1}{3}\%=33\frac{1}{3}$ times \$1500=\$50000=what the owners lost, it not being insured.
(20.) 1. 100%=\$100000,
2. $1\%=\frac{1}{100}$ of \$100000=\$1000, and
3. $1\frac{3}{4}\%=1\frac{3}{4}$ times \$1000=\$1750=what the owners paid the Mutual for insurance.
(21.) $\therefore$ \$50000 + \$1750 = \$51750=whole amount the owners lost.

III. $\therefore$ The owners of the building and stock lost \$51750.

EXAMPLES.

1. At $1\frac{3}{8}\%$, the premium for insuring my store was \$89.10; what was the amount of the insurance? *Ans.* \$6480.

2. The premium for insuring a tannery for $\frac{3}{4}$ of its value, at $1\frac{2}{3}\%$, was \$145.60; what was the value of the tannery? *Ans.* \$11648.

3. A store and its goods are worth \$6370. What sum must be insured, at 2%, to cover both property and premium? *Ans.* ——

4. The premium for insuring \$9870 was \$690.90; what was the rate? *Ans.* 7%.

5. A merchant whose stock of goods was valued at \$30000, insured it for $\frac{3}{4}$ of its value, at $\frac{3}{4}\%$. In a fire he saved \$5000 of the goods. What was his loss? What was the loss of the insurance companies? *Ans.* ——

6. A man paid \$180 for insuring his saw mill for $\frac{2}{3}$ of its value at 3%; what was the value of the mill? *Ans.* ——.

7. A house which has been insured for \$3500 for 10 years, at $\frac{2}{5}\%$ a year, was destroyed by fire; how much did the money received from the company exceed the cost of premiums? *Ans.* ——.

8. Took a risk on a house worth \$40000, at 2%; reinsured $\frac{1}{2}$ of it for $2\frac{1}{4}\%$, and $\frac{1}{4}$ of it at $2\frac{1}{2}\%$; in each case the amount covers premium; how much do I gain? *Ans.* \$99.558.

9. Took a risk at $1\frac{3}{4}\%$; reinsured $\frac{2}{3}$ of it at $1\frac{3}{4}\%$; my share of the premium was \$43; what was the amount of the risk? *Ans.* \$17200.

10. Took a risk at $2\frac{1}{4}\%$; reinsured $\frac{1}{2}$ of it at a rate equal to 3% of the whole, by which I lost \$37.50. What was the value of the risk? *Ans.* \$5000.

CHAPTER XIII.

INTEREST.

I. SIMPLE INTEREST.

1. ***Interest*** is money paid by the borrower to the lender for the use of money.

2. ***The Principal*** is the sum of money for which interest is paid.

3. ***The Rate*** of interest is the rate per cent. on \$1 for a certain time.

4. ***The Time*** is the period during which the money is on interest.

5. ***The Amount*** is the sum of the principal and interest.

6. ***Simple Interest*** is interest on the principal only.

7. ***Legal Interest*** is at the rate fixed by law.

8. ***Usury*** is interest at a rate greater than that allowed by law.

Let P=the principal,
r=the interest on \$1 for one year,
$R=1+r$=amount of \$1 for one year,
n=the number of years,
A=amount of P for n years,
Pr=simple interest on P for a year,
Pnr=simple interest on P for n years.
$P+Pnr=P\,(1+nr)$=amount of P for n years.
A=amount of P for n years.

$\therefore A=P+Pnr=P(1+mr)$. . . . (I.);

$\therefore P=\dfrac{A}{1+nr}$ . . . . (II.);

$\therefore Pnr=A-P$.

$\therefore P=\dfrac{A-P}{nr}=\dfrac{Interest}{nr}$ . . . . (III.);

$\therefore r=\dfrac{A-P}{Pn}$ . . . . (IV.); and

$\therefore n=\dfrac{A-P}{Pr}$ . . . . (V.).

When any three of the quantities A, P, n, r are given, the fourth may be found.

CASE I.

Given $\left\{\begin{array}{l}\text{Principal,}\\ \text{Rate, and}\\ \text{Time,}\end{array}\right\}$ to find the interest. Formula, $I=Prn$.

I. Find the interest of \$300 for two years at 6%.

By formula,

Interest $Prn = \$300 \times .06 \times 2 = \$36$.

By 100% method.

$$\text{II.}\begin{cases} 1.\ 100\% = \$300, \\ 2.\ 1\% = \frac{1}{100} \text{ of } \$300 = \$3, \text{ and} \\ 3.\ 6\% = 6 \text{ times } \$3 = \$18 = \text{interest for one year.} \\ 4.\ \$36 = 2 \text{ times } \$18 = \text{interest for 2 years.} \end{cases}$$

III. ∴ \$36=interest on \$300 at 6% for 2 years.

CASE II.

$$\text{Given}\begin{Bmatrix} \text{Principal,} \\ \text{Rate, and} \\ \text{Interest,} \end{Bmatrix}\text{to find the time.}\quad \text{Formula, } n = \frac{A-P}{Pr}.$$

I. In what time, at 5%, will \$60 amount to \$72?

By formula,

$$n = \frac{A-P}{Pr} = \frac{\$72-\$60}{\$60 \times .05} = 4 \text{ years.}$$

By 100% method.

$$\text{II.}\begin{cases} 1.\ \$72 = \text{amount.} \\ 2.\ \$60 = \text{principal.} \\ 3.\ \$72 - \$60 = \$12 = \text{interest for a certain time.} \\ 4.\ 100\% = \$60, \\ 5.\ 1\% = \frac{1}{100} \text{ of } \$60 = \$\frac{3}{5}, \text{ and} \\ 6.\ 5\% = 5 \text{ times } \$\frac{3}{5} = \$3 = \text{interest for one year.} \\ 7.\ \$12 = \text{interest for } 12 \div 3, \text{ or 4 years.} \end{cases}$$

III. ∴ \$60 at 5% will amount to \$72 in 4 years.

CASE III.

$$\text{Given}\begin{Bmatrix} \text{Principal,} \\ \text{Time, and} \\ \text{Interest,} \end{Bmatrix}\text{to find the rate.}\quad \text{Formula, } r = \frac{A-P}{Pn}.$$

I. I borrowed \$600 for two years and paid \$48 interest; what rate did I pay?

By formula,

$$r = \frac{A-P}{Pn} = \frac{I}{Pn} = \frac{\$48}{\$600 \times 2} = .04 = 4\%.$$

By 100% method.

$$\text{II.}\begin{cases} 1.\ \$48 = \text{interest for 2 years.} \\ 2.\ \$24 = \frac{1}{2} \text{ of } \$48 = \text{interest for 1 year.} \\ 3.\ \$600 = 100\%, \\ 4.\ \$1 = \frac{1}{600} \text{ of } 100\% = \frac{1}{6}\%, \text{ and} \\ 5.\ \$24 = 24 \text{ times } \frac{1}{6}\% = 4\%. \end{cases}$$

III. ∴ I paid 4% interest.

CASE IV.

Given $\left\{\begin{array}{l}\text{Time,}\\ \text{Rate, and}\\ \text{Interest.}\end{array}\right\}$ to find the principal. Formula, $P=\frac{A-P}{nr}=\frac{I}{nr}$.

I. The interest for 3 years, at 9%, is $21.60; what is the principal?

By formula,

$$P=\frac{A-P}{nr}=\frac{I}{nr}=\frac{\$21.60}{3\times.09}=\$80.$$

By 100% method.

II.
1. $21.60=interest for 3 years.
2. $7.20=$\frac{1}{3}$ of $21.60=interest for 1 year.
3. 100%=principal.
4. 9%=interest for 1 year.
5. $7.20=interest for 1 year.
6. ∴ 9%=$7.20,
7. 1%=$\frac{1}{9}$ of $7.20=$.80, and
8. 100%=100 times $.80=$80=principal.

III. ∴ $80=the principal.

CASE V.

Given $\left\{\begin{array}{l}\text{Time,}\\ \text{Rate, and}\\ \text{Amount}\end{array}\right\}$ to find the principal. Formula, $P=\frac{A}{1+nr}$.

I. What principal will amount to $936 in 5 years, at 6%?

By formula,

$$P=\frac{A}{1+nr}=\frac{\$936}{1+5\times.06}=\$720.$$

By 100% method.

II.
1. 100%=principal.
2. 6%=interest for 1 year.
3. 30%=5 times 6%=interest for 5 years.
4. 100%+30%=130%=amount.
5. $936=amount.
6. ∴ 130%=$936,
7. 1%=$\frac{1}{130}$ of $936=$7.20, and
8. 100%=100 times $7.20=$720=principal.

III. ∴ $720=the principal that will amount to $936 in 5 years at 6%.

I. In what time will any sum quadruple itself at 8%?

II.
1. 100%=principal. Then
2. 400%=the amount.
3. ∴ 400%—100%=300%=interest.
4. 8%=interest for 1 year.
5. 300%=interest for 300÷8, or $37\frac{1}{2}$ years.

III. ∴ Any principal will quadruple itself in $37\frac{1}{2}$ years at 8%.

II. TRUE DISCOUNT.

1. *Discount* on a debt payable by agreement at some future time, is a deduction made for "cash," or present payment; and arises from the consideration of the *present worth* of the debt.

2. *Present Worth* is that sum of money which, put on interest for the given time and rate, will amount to the debt at its maturity.

3. *True Discount* is the difference between the present worth and the whole debt.

Since P will amount to A in n years, P may be considered equivalent to A due at the end of n years.

∴ P may be regarded as the present worth of a given future sum A.

$$\therefore P=\frac{A}{1+nr}$$

I. Find the present worth of $590, due in 3 years, the rate of interest being 6%.

By formula,

$$P=\frac{A}{1+nr}=\frac{\$590}{1+3\times.06}=\$500.$$

By 100% method.

II.
1. 100%=present worth.
2. 6%=interest on present worth for 1 year.
3. 18%=3×6%=interest for 3 years.
4. 100%+18%=118%=amount, or debt.
5. $590=debt.
6. ∴ 118%=$590,
7. 1%=$\frac{1}{118}$ of $590=$5, and
8. 100%=100 times $5=$500=present worth.

III. ∴ $500=present worth of $590 due in 3 years at 6%.

I. A merchant buys a bill of goods amounting to \$2480; he can have 4 months credit, or 5% off for cash: if money is worth only 10% to him, what will he gain by paying cash?

II.
(1.) 100%=present worth of the debt.
(2.) 10%=interest on present worth for 1 year.
(3.) $3\frac{1}{3}$%=interest for 4 months.
(4.) 100%+$3\frac{1}{3}$%=$103\frac{1}{3}$%=amount of present worth, which equals the debt, by definition.
(5.) \$2480=the debt.
(6.) ∴ $103\frac{1}{3}$%=\$2480,
(7.) 1 %=$\frac{1}{103\frac{1}{3}}$ of \$2480=\$24, and
(8.) 100%=100 times \$24=\$2400=present worth.
(9.) \$2480—\$2400=\$80=true discount.
(10.) 1. 100%=\$2480.
2. 1%=$\frac{1}{100}$ of \$2480=\$24.80,
3. 5%=5 times \$24.80=\$124=trade discount, or discount for cash.
(11.) ∴ \$124—\$80=\$44=his gain by paying cash.

III. ∴ He would gain \$44 by paying cash.

(*R. 3d p., p. 258, prob. 10.*)

Remark.—It is clear that \$2480—\$124,=\$2356 would pay for the goods cash. If the merchant had this sum of money on hand, it would, in 4 months, at 10%, produce \$78.53$\frac{1}{3}$ interest. But if he pays his debt he will make \$124. Hence he will gain \$124—\$78.53$\frac{1}{3}$=\$45.46$\frac{2}{3}$.

III. BANK DISCOUNT.

1. *Bank Discount* is simple interest on the face of a note, calculated from the day of discount to the day of maturity, and paid in advance.

2. *The Proceeds* of a note is the amount which remains after deducting the discount from the face.

CASE I.

Given $\left\{\begin{array}{l}\text{Face of note,}\\ \text{Rate, and}\\ \text{Time,}\end{array}\right\}$ to find the discount and proceeds.

Formulæ, $\left\{\begin{array}{l}D=F\times r\times n\\ P=F-D.\end{array}\right.$

I. What is the bank discount of $770 for 90 days, at 6%?

By formula,

$$D=F\times r\times n=\$770\times .06\times\frac{(90+3)}{360}=\$11.935.$$

By 100% method.

II. $\begin{cases}1.\ 100\%=\$770,\\ 2.\ 1\%=\frac{1}{100}\text{ of }\$770=\$7.70,\text{ and}\\ 3.\ 6\%=6\text{ times }\$7.70=\$46.20=\text{discount for 1 year.}\\ 4.\ \$11.935=\frac{93}{360}\text{ of }\$46.20=\text{discount for 93 days.}\end{cases}$

III. ∴ $11.935=bank discount on $770 for 90 days at 6%.

CASE II.

Given $\left\{\begin{array}{l}\text{Proceeds,}\\ \text{Time, and}\\ \text{Rate,}\end{array}\right\}$ to find the face of the note.

Formula, $F=\frac{P}{1-rn}$.

I. For what sum must a note be made, so that when discounted at a bank, for 90 days, at 6% the proceeds will be $393.80?

By formula,

$$F=\frac{P}{1-rn}=\frac{\$393.80}{1-.06\times\frac{93}{360}}=\$400.$$

By 100% method.

II. $\begin{cases}1.\ 100\%=\text{face of the note.}\\ 2.\ 6\%=\text{discount for one year.}\\ 3.\ 1\frac{11}{20}\%=\frac{93}{360}\text{ of }6\%=\text{discount for 93 days.}\\ 4.\ 100\%-1\frac{11}{20}\%=98\frac{9}{20}\%=\text{proceeds.}\\ 5.\ \$393.80=\text{proceeds.}\\ 6.\ \therefore 98\frac{9}{20}\%=\$393.80,\\ 7.\ 1\%=\frac{1}{98\frac{9}{20}}\text{ of }\$393.80=\$4,\text{ and}\\ 8.\ 100\%=100\text{ times }\$4=\$400=\text{face of the note.}\end{cases}$

III. ∴ $400=face of the note.

CASE III.

Given rate of bank discount, to find the corresponding rate of interest.

Formula, rate of $I.=\frac{r}{1-rn}$.

I. What is the rate of interest when a 60 day note is discounted at 8% per annum?

By formula,

$$\text{rate of } I=\frac{r}{1-rn}=\frac{.08}{(1-\frac{63}{360}\times .08)}=.08\tfrac{56}{493}=8\tfrac{56}{493}\%.$$

By 100% method.

II.
1. 100% = face of note.
2. 8% = discount for 1 year.
3. $1\frac{2}{5}\% = \frac{63}{360}$ of 8% = discount for 63 days.
4. $100\% - 1\frac{2}{5}\% = 98\frac{3}{5}\%$ = proceeds.
5. $98\frac{3}{5}\% = 100\%$ of itself.
6. $1\% = \dfrac{1}{98\frac{3}{5}}$ of $100\% = \frac{500}{493}\%$, and.
7. $8\% = 8$ times $\frac{500}{493}\% = 8\frac{56}{493}\%$ = rate of interest.

III. ∴ The rate of interest on a 60 day note discounted at 8% per annum $= 8\frac{56}{493}\%$.

CASE IV.

Given the rate of interest, to find the corresponding rate of discount. Formula, $r = \dfrac{\text{rate of } I.}{1 + n \times \text{rate of } I.}$

I. What is the rate of discount on a 60 day note which yields 10% interest?

By formula,

$$r = \frac{\text{rate of } I.}{1 + n \times \text{rate of } I.} = \frac{.10}{1 + \frac{63}{360} \times .10} = .09\tfrac{337}{407} = 9\tfrac{337}{407}\%.$$

By 100% method.

II.
1. 100% = proceeds.
2. 10% = interest on proceeds for 1 year.
3. $1\frac{3}{4}\% = \frac{63}{360}$ of 10% = interest on proceeds for 63 days.
4. $100\% + 1\frac{3}{4}\% = 101\frac{3}{4}\%$ = face of note.
5. $101\frac{3}{4}\% = 100\%$ of itself.
6. $1\% = \dfrac{1}{101\frac{3}{4}}$ of $100\% = \frac{400}{407}\%$, and
7. $10\% = 10$ times $\frac{400}{407}\% = 9\frac{337}{407}\%$.

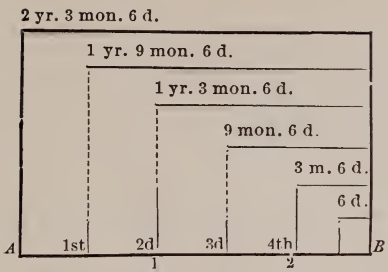
III. ∴ The rate of discount $= 9\frac{337}{407}\%$.

Note.—It must be borne in mind that the interest on the proceeds is equal to the discount on the face of the note.

IV. ANNUAL INTEREST.

1. ***Annual Interest*** is the simple interest of the principal and each year's interest from the time of its accruing until settlement.

I. No interest having been paid, find the amount due Sept. 7, 1877, on a note of $500, dated June 1, 1875, with interest at 6%, payable semi-annually.



1877	9	7
1875	6	1
2	3	6

II. (1.) $100\% = \$500$,

(2.) $1\% = \frac{1}{100}$ of $\$500 = \$5$, and

(3.) $6\% = 6$ times $\$5 = \$30$ = simple interest for 1 year.

(4.) $\$68 = 2\frac{4}{15}$ times $\$30$ = simple interest for 2 years, 3 months, 6 days.

(5.) $\$15 = \frac{1}{2}$ of $\$30$ = semi-annual interest.

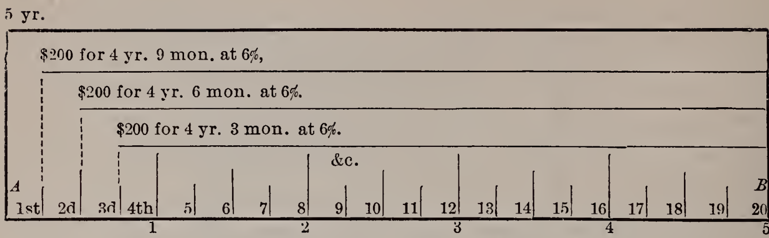
(6.)
1. $100\% = \$15$,
2. $1\% = \frac{1}{100}$ of $\$15 = \$.15$, and
3. $6\% = 6$ times $\$.15 = \$.90$ = interest on one semi-annual interest for 1 year.
4. $\$3.885 = 4\frac{19}{60}$ times $\$.90$ = interest on one semi-annual interest for the sum of the periods each draws int.

(7.) $\therefore$ $\$500 + \$68 + \$3.885 = \$571.885$ = amount of the note.

III. $\therefore$ $\$571.885$ = amount of the note.

Explanation.—At the end of six months there is $15 interest due; and, since it was not paid at that time, it drew interest from that time to the time of settlement, which is 1 yr. 9 mon. 6 da. At the end of the next six months, or at the end of the first year, there is another $15 due; and, since it was not paid at that time, it drew interest from that time to the time of settlement, which is 1 yr. 3 mon. 6 da. In like manner, the third semi-annual interest drew interest for 9 mon. 6 da., and the fourth for 3 mon. 6 da. This is the same as one semi-annual interest drawing interest for the sum of 1 yr. 9 mon. 6 da., 1 yr. 3 mon. 6 da., 9 mon. 6 da., 3 mon. 6 da. In the diagram, the line *A B* represents 2 yr. 3 mon. 6 day., *A* 1 represents the first year the note run, and 1-2 represents the second year the note run. Between *A* and 1 is a small mark that denotes the semi-annual period; also one between 1 and 2. By such diagrams, the time for which to compute interest on the simple interest may be easily found.

I. The interest of U. S. 4% bonds is payable quarterly in gold; granting that the income from them might be immediately invested, at 6%, what would the income on 20 1000-dollar bonds amount to in 5 years, with gold at 105?



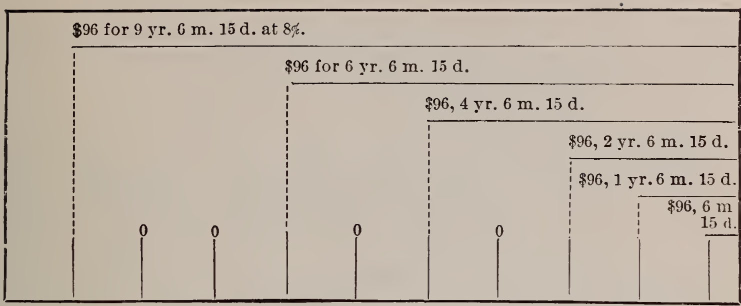
II.
(1.) \$1000=par value of one bond.
(2.) \$20000=par value of 20 bonds.
(3.) 100%=\$20000,
(4.) 1%=$\frac{1}{100}$ of \$20000=\$200, and
(5.) 4%=4 times \$200=\$800=income for one year.
(6.) \$4000=5 times \$800=income for five years.
(7.) \$200=$\frac{1}{4}$ of \$800=interest due at the end of first quarter, and which draws interest to time of settlement.
(8.)
1. 100%=\$200,
2. 1%=$\frac{1}{100}$ of \$200=\$2, and
3. 6%=6 times \$2=\$12=interest on quarterly interest for one year.
4. \$570=47$\frac{1}{2}$ times \$12=interest on quarterly interest for the sum of 4$\frac{3}{4}$ yr.+4$\frac{1}{2}$ yr.+4$\frac{1}{4}$ yr.+. . . . +$\frac{1}{4}$ yr., or 47$\frac{1}{2}$ years.

(9.) ∴ \$4000+\$570=\$4570=income of bonds in gold.
(10.) \$1.00 in gold=\$1.05 in currency.
(11.) \$4570 in gold=4570 times \$1.05=\$4798.50 in currency.

III. ∴ The bonds yield \$4798.50 in currency.

Explanation.—It must be borne in mind that the quarterly interest, \$200, is put on interest at 6% as soon as it is due. At the end of the first quarter there is \$200 due which draws interest at 6% for the remaining time, 4 years, 9 months. The second quarterly interest is due at the end of six months and draws interest for the remaining time, 4 years 3 months, and so on with the remaining quarterly payments. This is the same as one quarterly payment drawing interest for the sum of 4$\frac{3}{4}$ yr.+4$\frac{1}{2}$ yr. +4$\frac{1}{4}$ yr.+etc., or 47$\frac{1}{2}$ years.

I. What was due on a note of \$1200, dated January 16, 1882, and due July 1, 1892, and bearing interest at 8%, payable annually, if the 2, 3, 5, and 7th years' interest were paid?



II.
(1.) 100%=$1200.
(2.) 1%=$\frac{1}{100}$ of $1200=$12, and
(3.) 8%=8 times $12=$96=simple int. for one year.
(4.) $480=5×$96=five simple interests.
(5.) $48=$\frac{1}{2}$ of $96=interest for 6 months.
(6.) $4=$\frac{1}{12}$ of $48=interest for 15 days.
(7.) ∴ $532=simple interest unpaid.
(8.)
1. 100%=$96.
2. 1%=$\frac{1}{100}$ of $96=$.96, and [simple interest.
3. 8%=8 times $.96=$7.68=interest on one year's
4. $193.92=25$\frac{1}{4}$ times $7.68=interest on year's simple interest for 9 yr. 6 mon. 15 da.,+6 yr. 6 mon. 15 da.,+4 yr. 6 mon. 15 da.,+2 yr. 6 mon. 15 da., +1 yr. 6 mon. 15 da.,+6 mon. 15 da., or 25 yr. 3 mon.

(9.) ∴ $532+$193.92=$725.92=amount of interest due. [1, 1892.
(10) $1200+725.92=$1925.92=amount due July

III. ∴ $1925.92=whole amount due July 1, 1892.

V. COMPOUND INTEREST.

1. ***Compound Interest*** is interest on a principal formed by adding interest to a former principal.

Let P=principal on compound interest.
r=rate,
$R=(1+r)$=amount of one dollar for 1 year.
$P\,(1+r)=PR$=amount of P dollars for 1 year.
$P\,(1+r)^2=PR^2$=amount of P dollars for 2 years.
$P\,(1+r)^3=PR^3$=amount of P dollars for 3 years.
$P\,(1+r)^n=PR^n$=amount of P dollars for n years.

Let A=amount of P dollars in n years, and
I=the compound interest of P dollars for n years.

Then $I=PR^n-P$. . . . . I.
$A=PR^n$. . . . . II.
$\therefore\ P=\frac{A}{R^n}$. . . . . III.

$\therefore\ R=\sqrt[n]{\frac{A}{P}}$.......IV. Applying logarithms to $R^n=\frac{A}{P}$,

n log. R=log. A—log. P, whence

$n=\frac{\text{log. } A-\text{log. } P}{\text{log. } R}$... V.

When compound interest is payable semi-annually.

$P\ (1+\frac{r}{2})$=amount of P dollars for $\frac{1}{2}$ year.

$P\ (1+\frac{r}{2})^2$=amount of P dollars for 1 year.

$P\ (1+\frac{r}{2})^{2n}$=amount of P dollars for n years.

$\therefore\ A=P\ (1+\frac{r}{2})^{2n}$, when payable semi-annually.

When compound interest is payable quarterly,

$P\ (1+\frac{r}{4})$=amount of P dollars for $\frac{1}{4}$ year.

$P\ (1+\frac{r}{4})^2$=amount of P dollars for $\frac{1}{2}$ year.

$P\ (1+\frac{r}{4})^3$=amount of P dollars for $\frac{3}{4}$ year.

$P\ (1+\frac{r}{4})^4$=amount of P dollars for 1 year.

$P\ (1+\frac{r}{4})^{4n}$=amount of P dollars for n years.

$\therefore\ A=P\ (1+\frac{r}{4})^{4n}$.

When the interest is payable monthly,

$A=P\ (1+\frac{r}{12})^{12n}$.

When the interest is payable q times a year,

$A=P\ (1+\frac{r}{q})^{qn}$.

CASE I.

Given $\left\{\begin{array}{l}\text{Principal,}\\ \text{Rate, and}\\ \text{Time,}\end{array}\right\}$ to find the compound interest and amount.

Formulæ, $\left\{\begin{array}{l}I=PR^n-P,\\ A=PR^n.\end{array}\right.$

I. Find the compound interest and amount of \$500 for 3 years at 6%.

By formulæ,

$A=PR^n=\$500\times(1+.06)^3=\$595.508$, and

$I=PR^n-P=\$500\times(1+.06)^3-\$500=\$95.508$.

Remark.—In compound interest, the 100% method becomes very tedious.

By 100% method.

II.
(1.) 100%=\$500,
(2.) 1%=$\frac{1}{100}$ of \$500=\$5,
(3.) 6%=6 times \$5=\$30=interest for 1 year.
(4.) \$500+\$30=\$530=amount, or principal for the second year.
(5.)
1. 100%=\$530,
2. 1%=$\frac{1}{100}$ of \$530=\$5.30,
3. 6%=6 times \$5.30=\$31.80=interest for second year.
4. \$530+\$31.80=\$561.80=amount, or principal for the third year.

(6.) 1. 100%=\$561.80,
2. $1\% = \frac{1}{100}$ of \$561.80=\$5.618, and
3. 6%=6 times \$5.618=\$33.708=interest for third year.
4. 561.80+\$33.708=\$595.508=amount at end of the third year.

(7.) \$595.508—\$500=\$95.508=compound interest.

III. ∴ \$95.508=compound interest, and
\$595.508=compound amount.

CASE II.

Given Principal, Rate, and Compound Interest, to find the time.

Formula, $n = \frac{\log. A - \log. P}{\log. R}$.

I. In what time will \$8000 amount to \$12000, at 6% compound interest?

By formula,

$$n = \frac{\log. A - \log. P}{\log. R} = \frac{\log. 12000 - \log. 8000}{\log. 1.06} = \frac{4.079181 - 3.903090}{.025306} = 6 \text{ yr. } 11 \text{ mon. } 15 \text{ da.}$$

We may solve the problem thus: $\$8000(1.06)^n = \$12000$, whence $(1.06)^n = 12000 \div 8000 = 1.50$. Referring to a table of compound amounts and passing down the column of 6%, we find this amount between 6 years and 7 years.

The amount for 6 years is 1.4185191; the amount for required time is 1.50. ∵ There is a difference of 1.50—1.4185191, or .0814809. The difference for the year between 6 and 7 is .0851112. .0851112=amount for the whole period between 6 and 7, .0814809=amount for $\frac{814809}{851112}$ of the period or, 11 mon. 15 da. ∴ The whole time=6 yr. 11 mon. 15 da.

CASE III.

Given Principal, Compound Intesest or Amount, and Time, to find the rate.

Formula, $r = \sqrt[n]{\frac{P}{A}} - 1$.

I. At what rate, by compound interest, will \$1000 amount to \$1593.85 in 8 years?

By formula,

$$r = \sqrt[n]{\frac{A}{P}} - 1 = \sqrt[n]{\frac{\$1593.85}{\$1000}} - 1 = .06 = 6\%.$$

Thru formula ?

CASE IV.

Given $\left\{\begin{array}{l}\text{Compound Interest or Amount}\\ \text{Time, and}\\ \text{Rate,}\end{array}\right\}$ to find the principal.

$$\text{Formulæ, } P=\begin{cases}\dfrac{A}{R^n}, \text{or}\\[2ex] \dfrac{I}{R^n-1}\end{cases}$$

I. What principal, at compound interest will amount to 27062.85 in 7 years at 4%?

By formula,

$$P=\frac{A}{R^n}=\frac{27062.85}{(1.04)^7}=\$20565.54$$

CHAPTER XIV.

ANNUITIES.

1. ***An Annuity*** is a sum of money payable at yearly, or other regular intervals.

2. ***Annuities*** $\begin{cases}\text{1. Perpetual, or}\\ \text{2. Limited;}\\ \text{3. Certain, or}\\ \text{4. Contingent.}\end{cases}$

3. ***A Perpetual Annuity*** is one that continues forever.

4. ***A Limited Annuity*** ceases at a certain time.

5. ***A Certain Annuity*** begins and ends at fixed times.

6. ***A Contingent Annuity*** begins or ends with the happening of a contingent event.

7. ***An Immediate Annuity*** is one that begins at once.

8. ***A Deferred Annuity*** is one that does not begin immediately.

9. ***The Final or Forborne*** value of an annuity is the amount of the whole accumulated debt and interest, at the time the annuity ceases.

10. ***The Present Value*** of an annuity is that sum, which, put at interest for the given time and given rate, will amount to the initial value.

11. ***The Initial Value*** of an annuity is the value of a deferred annuity at the time it commences.

CASE I.

Given $\left\{\begin{array}{l}\text{Annuity,}\\ \text{Time, and}\\ \text{Rate,}\end{array}\right\}$ to find the initial value of a perpetuity.

I. What is the initial value of a perpetual annuity of $300 a year, allowing interest at 6%?

II.
1. 100%=initial value.
2. 6%=interest for 1 year.
3. $300=interest for 1 year.
4. ∴ 6%=$300.
5. 1%=$\frac{1}{6}$ of $300=$50, and
6. 100%=100 times $50=$5000=initial value.

III. ∴ Initial value=$5000. (*R. H. A., p. 310, prob. 1.*)

I. What is the initial value of a perpetual leasehold of $2500 a year payable quarterly, interest payable semi-annually at 6%; 6% payable annually; 6% payable quarterly?

II.

A.
1. Let S=the annuity. Then S=the amount due in 3 months.
2. $S+S(1+\frac{r}{4})$=amount due in 6 months.
3. ∴ $A=S+S(1+.01\frac{1}{2})$=$625+$625 $(1.01\frac{1}{2})$= 1259.37\frac{1}{2}$=amount due at the end of 6 months.
4. 100%=initial value.
5. 3%=semi-annual annuity.
6. 1259.37\frac{1}{2}$=semi-annual annuity.
7. ∴ 3%=1259.37\frac{1}{2}$.
8. 1%=$\frac{1}{3}$ of 1259.37\frac{1}{2}$=$419.7916$\frac{2}{3}$, and
9. 100%=100 times 419.7916\frac{2}{3}$=initial value.

B.
1. Let S=amount due in 3 months. Then
2. $S+S(1+\frac{r}{4})$=amount due in 6 months, [and
3. $S+S(1+\frac{r}{4})+S(1+\frac{2r}{4})$=amount due in 9 months,
4. $S+S(1+\frac{r}{4})+S(1+\frac{2r}{4})+S(1+\frac{3r}{4})$= amount due in 1 year. [$(1+\frac{.18}{4})$=$2556.25.
5. ∴ A=$625+$625$(1+\frac{.06}{4})$+ 625(1+\frac{.12}{4})$ + $625-
6. 100%=initial value.
7. 6%=annuity.
8. $2556.25=annuity.
9. ∴ 6%=$2556.25.
10. 1%=$\frac{1}{6}$ of $2556.25=$426.0416$\frac{2}{3}$, and [value.
11. 100%=100 times 426.0416\frac{2}{3}$=$42604.16$\frac{2}{3}$=initial

C.
1. 100%=initial value.
2. 1$\frac{1}{2}$%=quarterly annuity.
3. $625=quarterly annuity.
4. ∴ 1$\frac{1}{2}$%=$625.
5. 1%=$\frac{1}{1\frac{1}{2}}$ of $625=$416.6666$\frac{2}{3}$, and
6. 100%=100 times 416.6666\frac{2}{3}$=$41666.66$\frac{2}{3}$.

III. $\therefore \left\{ \begin{array}{l} \text{Initial value of A}=\$41979.16\frac{2}{3}, \\ \text{Initial value of B}=\$42604.16\frac{2}{3}, \text{ and} \\ \text{Initial value of C}=\$41666.66\frac{2}{3}. \end{array} \right.$

(*R. H. A., p. 310, prob. 5.*)

CASE II.

Given $\left\{ \begin{array}{l} \text{Annuity,} \\ \text{Interval,} \\ \text{Rate, and} \\ \text{Time the perpetuity is deferred,} \end{array} \right\}$ to find the present value of a deferred perpetuity.

Let S=the annuity, r=the rate, and $R=1+r$. Then by Case I., the initial value of S is $S\div r$. To find the present value of the initial value, we use formula III., compound interest. $\therefore P = \frac{S}{r(1+r)^t} = \frac{S}{R^t(R-1)}$ in which t is the time the perpetuity is deferred.

I. Find the present value of a perpetuity of \$250 a year, deferred 8 years, allowing 6% interest.

By formula,

$$P=\frac{S}{R^t(R-1)}=\frac{\$250}{(1+.06)^8(1+.06-1)}=\frac{\$250}{.06(1.06)^8}=\$2614.22.$$

By 100% method.

II. (1.) 100%=initial value.
(2.) 6%=annuity.
(3.) \$250=annuity.
(4.) $\therefore$ 6%=\$250.
(5.) 1%=$\frac{1}{6}$ of \$250=\$41$\frac{2}{3}$, and
(6.) 100%=100 times \$41$\frac{2}{3}$=\$4166.66$\frac{2}{3}$=initial value.
(7.)
1. 100%=present value of \$4166.66$\frac{2}{3}$ due in 8 years at 6%.
2. 159.38481%=$(1.06)^8\times$100%=compound amount of the present value for 8 yr. at 6%.
3. $\therefore$ 159.38481%=\$4166.66$\frac{2}{3}$,
4. 1%=$\frac{1}{159.38481}$ of \$4166.66$\frac{2}{3}$=\$26.1422, and
5. 100%=100 times \$26.1422 = \$2614.22 = present value.

III. $\therefore$ The present value of a perpetuity of \$250 a year deferred 8 years at 6% interest=\$2614.22.

I. Find the present value of an estate which, in 5 years, is to pay \$325 a year forever; interest 8%, payable semi-annually.

By formula,

$$P=\frac{S}{[(1+\frac{r}{2})^2-1]R^t}=\frac{\$325}{[(1.04)^2-1](1.08)^5}=\frac{\$325}{.0816(1.08)^5}=$$
\$2690.67.

By 100% method.

II. (1.) 100%=initial value.
(2.) 4%=amount due in 6 months.
(3.) 4%+(1.04)×4%=8.16%=amount due in 1 year.
(4.) $325=amount due in 1 year.
(5.) ∴ 8.16%=$325,
(6.) 1%=$\frac{1}{8.16}$ of $325=$39.828431, and
(7.) 100%=100 times $39.828431= $3982.8431 = initial value.
(8.) 1. 100%=present value of $3982.8431.
2. 146.93281%=$(1.08)^5$×100%=compound amount of 100% for 5 yr. at 8%.
3. ∴ 146.93281%=$3982.8431,
4. 1%=$\frac{1}{146.93281}$ of $3982.8431=$26.9067, and
5. 100% = 100 times $26.9067 = $2690.67 = present value.

III. ∴ $2690.67=present value of the estate.

(*R. H. A.*, *p.* 311, *prob.* 4.)

Explanation.—The initial value is a sum of money which placed on interest at 8% payable semi-annually will produce $325 per year. But 8% payable semi-annually is the same as 8.16% payable annually. Hence 8.16% is the annual payment. But $325 is the annual payment. Hence 8.16%=$325, from which we find that $3982.8431 is the initial value, or the amount that will produce $325 per year. Then the present value of a sum of money that will pay $325 is $3982.8431 if the payments are to begin at once, but $3982.8431÷$(1.08)^5$ if the payments are not to begin until the end of 5 years.

CASE III.

Given Rate, Annuity, Time to run, and Interval, to find the present valve of an annuity certain.

(*a*) Let P denote the present value. The amount of P for n years$=PR^n=A$.

Let S=the payment, or amount due the first year.

$S+SR$=the amount due the second year.

$S+SR+SR^2$=the amount due the third year.

$S+SR+SR^2+SR^3$ = the amount due the fourth year.

$S+SR+SR^2+SR^3+\ldots\ldots\ldots+SR^{n-1}$ = amount due the nth year.

$\therefore A=S+SR+SR^2+SR^3+\ldots\ldots+SR^{n-1}\ldots\ldots\ldots(1)$

$AR=SR+SR^2+SR^3+SR^4+\ldots\ldots SR^n\ldots\ldots(2)$, by multiplying (1) by R.

$AR-A=SR^n-S\ldots(3)$, by subtracting (1) from (2).

$\therefore A=\frac{S(R^n-1)}{R-1}\ldots\ldots(4.)$ But $PR^n=A$.

$$\therefore\ PR^n=\frac{S(R^n-1)}{(R-1)}\ \ldots\ldots (5.),\text{ whence}$$

$$P=\frac{S(R^n-1)}{R^n(R-1)}=\frac{S}{R-1}\times\frac{R^n-1}{R^n}\ \ldots\ldots (6).$$

(*b.*) When the annuity is to begin at a certain time, and then to continue a certain time.

Let p=the number of years the annuity is deferred, and q= the number of years the annuity continues. Then

$P'=\frac{S}{R-1}\times\frac{R^{p+q}-1}{R^{p+q}}$=the present value of an annuity S, for the time $(p+q)$ years, and

$P''=\frac{S}{R-1}\times\frac{R^p-1}{R^p}$=the present value of an annuity S, for p years.

$$\therefore\ P=P'-P''=\frac{S}{R-1}\times\frac{R^{p+q}-1}{R^{p+q}}-\frac{S}{R-1}\times\frac{R^p-1}{R^p}=\frac{S}{R-1}\left(\frac{R^{p+q}-1}{R^{p+q}}-\frac{R^p-1}{R^p}\right)=\frac{S}{R-1}\left[1-\frac{1}{R^{p+q}}-\left(1-\frac{1}{R^p}\right)\right]=\frac{S}{R-1}\left(\frac{1}{R^p}-\frac{1}{R^{p+q}}\right)=\frac{S}{R-1}\times\frac{R^q-1}{R^{p+q}}\ \ldots (7.)$$

I. Find the present value of an annuity of \$250, payable annually for 30 years at 5%.

Given S, n, and r.

By formula,

$$P=\frac{S}{R-1}\times\frac{R^n-1}{R^n}=\frac{\$250}{.05}\times\frac{(1.05)^{30}-1}{(1.05)^{30}}=\$3843.1135.$$

By 100% method.

II.
- (1.) 100%=initial value.
- (2.) 5%=annuity.
- (3.) \$250=annuity.
- (4.) ∴ 5%=\$250,
- (5.) 1%=$\frac{1}{5}$ of \$250=\$50, and
- (6.) 100%=100 times \$50%=\$5000=initial value of an *immediate* perpetuity of \$250 per year.
- (7.)
  1. 100%=present value of an annuity *deferred* 30 years.
  2. 432.19424%=$(1.05)^{30}\times100\%$=amount of present value for 30 years.
  3. ∴ 432.19424%=\$5000,
  4. 1%=$\frac{1}{432.19424}$ of \$5000=\$11.568865, and
  5. 100%=100 times \$11.568865=\$1156.8865=present value of annuity of \$250 deferred 30 years.
- (8.) ∴ \$5000—\$1156.8865=\$3843.1135=present value of an annuity continuing 30 years.

III. ∴ \$3843.1135=present value of an annuity of \$250, payable annually for 30 years,

Remark.—Since \$5000 is the initial value which, in this case, is also the present value of an immediate perpetual annuity, or perpetuity of \$250, and \$1156.8865 the present value of an annuity of \$250 deferred 30 years, \$5000—\$1156.8865=\$3843.1135= the present value of an annuity of \$250 continuing for 30 years at 5%.

I. Find the present value of an annuity of \$826.50, to commence in 3 years and run 13 years, 9 months, interest 6%, payable semi-annually.

Given S=\$826.50, r=.06, p=3 years, and $q=13\frac{3}{4}$ years.

When interest is payable semi-annually, $R=(1+\frac{r}{2})^2$.

By formula (7),

$$P=\frac{S}{R-1}\times\frac{R^q-1}{R^{(p+q)}}=\frac{\$826.50}{.0609}\times\frac{(1.0609)^{13\frac{3}{4}}-1}{(1.0609)^{16\frac{3}{4}}}=\$6324.69.$$

By 100% method.

II.
(1.) 100%=initial value.
(2.) 3%=amount due in 6 months.
(3.) 3%+3% (1.03)=6.09%=amount due in 1 year.
(4.) \$826.50=amount due in 1 year.
(5.) ∴ 6.09%=\$826.50,
(6.) 1%=$\frac{1}{6.09}$ of \$826.50=\$135.712643, and
(7.) 100% = 100 times \$135.712643 = \$13571.2643= initial value
(8.)
1. 100%=present value of a perpetuity of \$826.50 deferred 3 years.
2. 119.40523%=(1.0609)^2 times 100%=amount of present value for 3 years.
3. ∴ 119 40523%=\$13571.2643,
4. 1%=$\frac{1}{119.40523}$ of \$13571.2643=\$113.6586,
5. 100%=100 times \$113.6586=\$11365.86=present value of such a perpetuity deferred 3 years

(9.)
1. 100%=present value of such a perpetuity deferred $16\frac{3}{4}$ years.
2. 269.212027%=(1.0609)$^{16\frac{3}{4}}$ times 100% =amount of present value for $16\frac{3}{4}$ years
3. ∴ 269.212027%=\$13571.2643,
4. 1%=$\frac{1}{269.212027}$ of \$13571.2643=\$50.4117,
5. 100%=100 times \$50.4117 = \$5041.17 = present value of such a perpetuity deferred $16\frac{3}{4}$ years.

(10.) ∴ \$11365.86—\$5041.17=\$6324.69=present value of an annuity of \$826.50 deferred 3 years and continuing $13\frac{3}{4}$ years.

III. ∴ \$6324.69=present value of \$826.50, etc.

If the annuity is to begin in p years and continue forever, the formula,

$P=\frac{S}{R-1}\times\frac{R^q-1}{R^{p+q}}$ becomes $P=\frac{S}{R^p(R-1)}$.

For, since $P=\frac{S}{R-1}\left[\left(1-\frac{1}{R^{p+q}}\right)-\left(1-\frac{1}{R^p}\right)\right]$, if $q=\infty$, the quantity

$1-\frac{1}{R^{p+q}}=1-\frac{1}{R^{p+\infty}}=1-\frac{1}{\infty}=1-0$, approaches 1 as its limit,

and we have $P=\frac{S}{R-1}\left[\left(1-0\right)-\left(1-\frac{1}{R^p}\right)\right]=\frac{S}{(R-1)R^p}$.

I. Find the present value of a perpetual annuity of $1000 to begin in 3 years, at 4% interest.

By formula,

$P=\frac{S}{(R-1)R^p}=\frac{\$1000}{.04\times(1.04)^3}=\$22224.92=$ present [value of the annuity.

By 100% method.

II.
- (1.) 100%=initial value.
- (2.) 4%=annuity.
- (3.) $1000=annuity.
- (4.) ∴ 4%=$1000,
- (5.) 1%=¼ of $1000=$250, and
- (6.) 100%=100 times $250=$25000=initial value of [$1000.
- (7.)
  1. 100%=present value.
  2. 112.4864% = $(1.04)^3$ times 100% = amount of present value for 3 years at 4%.
  3. ∴ 112.4864%=$25000,
  4. 1%=$\frac{1}{112.4864}$ of $25000=$222.2492, and
  5. 100%=100 times $222.2492=$22224.92=present value.

III. ∴ $22224.92=present value of an annuity of $1000 to begin in 3 years at 4%.

CASE IV.

Given {Annuity, Rate, Interval, and Time to run,} to find the final or forborne value.

Let S=amount due first year.

$S+SR$=amount due second year.

$S+SR+SR^2$=amount due third year.

$S+SR+SR^2+SR^3$=amount due the fourth year.

$S+SR+SR^2+SR^3+\ldots\ldots+SR^{n-1}$= amount due the nth year.

Let A=amount due the nth year.

$\therefore A=S+SR+SR^2+SR^3+\ldots\ldots+SR^{n-1}\ldots(1)$.

$AR = SR + SR^2 + SR^3 + SR^4 + \ldots\ldots + SR^n \ldots\ldots$ (2), by multiplying (1) by R. [from (2).

$\therefore AR - A = SR^n - S \ldots\ldots$ (3), by subtracting (1)

$\therefore A = \dfrac{S(R^n - 1)}{R - 1} \ldots\ldots$ (4.)

I. A pays \$25 a year for tobacco; how much better off would he have been in 40 years if he had invested it at 10% per annum?

By formula,

$A = \dfrac{S}{R-1} \times (R^n - 1) = \dfrac{\$25}{.10} \times [(1.10)^{40} - 1] = \$11064.8139.$

By 100% method.

II.
1. 100%=initial value.
2. 10%=annuity.
3. \$25=annuity.
4. ∴ 10%=\$25,
5. 1%=$\frac{1}{10}$ of \$25=\$2.50, and
6. 100%=100 times \$2.50=\$250=initial value.
7. \$44.2592556=$[(1.10)^{40}-1]\times$\$1=compound interest of \$1 for 40 yr. at 10%.
8. ∴ \$11064.8139=44.2592556×\$250=compound int. of \$250 for 40 yr. at 10%.

III. ∴ He would be \$11064.8139 better off.

Remark.—\$250 placed on interest at 10% will produce \$25 per year. If this interest be put on interest at 10%, instead of spending it for tobacco, it will amount to \$11064.8139 in 40 years. This would be a very sensible and profitable investment for every young man to make, who is a slave to the pernicious habit.

I. An annuity, at simple interest 6%, in 14 years, amounted to \$116.76; what would have been the difference, had it been at compound interest 6%?

II.
(1.) 100%=initial value, or the principal that would produce the annuity.
(2.) 6%=annuity for 1 year.
(3.) 84%=14×6%=annuity for 14 years.
(4.)
1. 100%=6%,
2. 1%=$\frac{1}{100}$ of 6%=$\frac{3}{50}$%, and
3. 6%=6 times $\frac{3}{50}$%=$\frac{9}{25}$%=interest on annuity for 1 year.
4. 32.76%=91 times $\frac{9}{25}$%=interest on annuity for (1+2+3+ ..... +14), or 91 years.

(5.) 84%+32.76%=116.76%=whole amount of the annuity.
(6.) \$116.76=whole amount of the annuity.
(7.) ∴ 116.76%=\$116.76,
(8.) 1%=$\frac{1}{116.76}$ of \$116,76=\$1, and
(9.) 100%=100 times \$1=\$100=initial value.
(10.) 6%=6 times \$1=\$6=annuity.

(11.) $\$1.260904=[(1.06)^{14}-1]\times\$1=$ compound interest on \$1 for 14 yrs. at 6%.
(12.) $\$126.0904=1.260904\times\$100=$ compound interest on \$100 for 14 yrs. at 6%.
(13.) $\therefore$ \$126.0904—\$116.76=\$9.3304=difference.

III. $\therefore$ The difference=\$9.3304.

CASE V.

Given $\left\{\begin{array}{l}\text{Final Value or Present Value}\\ \text{Rate, and}\\ \text{Time to run,}\end{array}\right\}$ to find the annuity.

Solving $P=\dfrac{S}{R-1}\times\dfrac{R^n-1}{R^n}$ with respect to S and we have $S=\dfrac{P(R-1)R^n}{R^n-1}=rP\times\dfrac{R^n}{R^n-1}$ . . . . . . (1). If A=the final or forborne value, by the formula in the last case, we have $A=\dfrac{S}{R-1}\times R^n-1$. Solving this with respect to S, we have.

$$S=\frac{(R-1)A}{R^n-1}=\frac{rA}{R^n-1}\;\ldots\ldots\;(2).$$

I. How much a year should I pay, to secure \$15000 at the end of 17 years, interest 7%?

By formula (2),

$$S=\frac{rA}{R^n-1}=\frac{.07\times\$15000}{(1.07)^{17}-1}=\$486.38.$$

By 100% method.

II.
(1.) 100%=annuity.
(2.) 7%=annuity.
(3.) $\therefore$ 7%=100%,
(4.) $1\%=\frac{1}{7}$ of $100\%=14\frac{2}{7}\%$, and
(5.) 100%=100 times $14\frac{2}{7}\%=1428\frac{4}{7}\%$=initial value.
(6.)
1. 100%=present value of $1428\frac{4}{7}\%$ due in 17 years.
2. 315.8815%=amount of present value for 17 years.
3. $\therefore$ $315.8815\%=1428\frac{4}{7}\%$,
4. $1\%=\frac{1}{315.8815}$ of $1428\frac{4}{7}\%=4.522591\%$, and
5. 100%=100 times 4.522591%=452.2591%=present value.

(7.) $\therefore$ $1428\frac{4}{7}\%-452.2591\%=976.3223\%$ = present value of an annuity running 17 years.
(8.) $3.1588152\%=(1.07)^{17}$ times 1%=amount of 1% for 17 years.
(9.) $3084.0217\%=(1.07)^{17}$ times 976.3223%=amount of 976.3223% for 17 years at 7%.
(10.) \$15000=amount, or final value.
(11.) $\therefore$ 3084.0217%=\$15000.
(12.) $1\%=\frac{1}{3084.0217}$ of \$15000=\$4.8638, and
(13.) $100\%=100\times\$4.8638=\$486.38$=annuity.

III. $\therefore$ I must pay \$486.38.

CASE VI.

Given $\left\{\begin{array}{l}\text{Annuity,}\\ \text{Present Value of the Annuity, and}\\ \text{Rate,}\end{array}\right\}$ to find time it runs.

In formula (6), Case III., we have $P=\frac{S}{R-1}\times\frac{R^n-1}{R^n}$, whence $\frac{R^n-1}{R^n}=\frac{P(R-1}{S}$, or $1-\frac{1}{R^n}=\frac{Pr}{S}$, $\frac{1}{R^n}=1-\frac{Pr}{S}=\frac{S-Pr}{S}$.

$\therefore R^n=\frac{S}{S-Pr}$ . . . . . (1). Applying logarithms,

n log. $R=$log. $\left(\frac{S}{S-Pr}\right)$.

$\therefore$ $n=$log. $\left(\frac{S}{S-Pr}\right)\div$log. $R=\frac{\text{log. } S-\text{log. } (S-Pr)}{\text{log. } R}$. . . . . (2).

I. In how many years can a debt of \$1,000,000, drawing interest at 6%, be discharged by a sinking fund of \$80,000 per year?

By formula (2),

$$n=\frac{\text{log.} S-\text{log.}(S-Pr)}{\text{log. } R}=\frac{\text{log. } 80000-\text{log } (80000-1000000\times.06)}{\text{log. } 1.06}$$

$$=\frac{\text{log. } 80000-\text{log. } 20000}{\text{log. } 1.06}=\frac{4.903090-4.301030}{.025306}=\frac{.602060}{.025306}=23.857$$

years.

By another method.

Assume \$1,000,000 to be the present value of an annuity of \$80000 a year. Then \$12.50 may be considered as the present value of \$1 for the same time and rate. By reference to a table of present worths \$12.50, which is 1000000÷80000, will be found to be between 23 and 24 years.

Note.—A table of present worths may be computed by formula (6.), Case III., in which put $S=\$1$.

I. In what time will a debt of \$10000, drawing interest at 6%, be paid by installments of \$1000 a year.

By formula,

$$n=\frac{\text{log. } S-\text{log. } (S-Pr)}{\text{log. } R}=\frac{\text{log. } 1000-\text{log.}(1000-10000\times.06)}{\text{log. } 1.06}$$

$$=\frac{3-2.602060}{.025306}=15.725 \text{ years}=15 \text{ yr. } 8 \text{ mo. } 21 \text{ da.}$$

By another method.

Assume \$10000 to be the present value of an annuity of \$1000 a year. Then \$10000÷1000=\$10=the present value of \$1 for the same time and rate. By referring to a table of present worth we find this amount between 15 and 16 years. $\therefore$ The time is 15 years +

The compound amount of \$10000 for 15 yr. at 6%=	\$23965.58
The final value of \$1000 for 15 years at 6%=	\$23275.97
Balance=	\$ 689.61

This balance, \$689.61, will require a fraction of a year to discharge it. The part of a year required, will be such a fraction of a year as the amount of \$689.61 for the *fraction* of a year is of \$1000.

6% of \$689.61 for the *fraction* of a year=\$41.3766× *fraction* of a year.

$\therefore$ \$689.61+\$41.3766× *fraction* of a year=the amount of \$689.61 for the *fraction* of a year. This amount divided by \$1000, a yearly payment, will give the *fraction*.

$$\therefore \frac{\$689.61+\$41.3766\times fraction}{\$1000}=fraction, \text{ whence}$$

$$\$689.61+\$41.3766\times fraction=\$1000\times fraction$$

$$\therefore \$1000\times fraction-\$41.3766\times fraction=\$689.61, \text{ or}$$

$$\therefore \$958.628\times fraction=\$689.61.$$

$$\therefore fraction=\frac{689.61}{958.628}=8 \text{ months, 19 days.}$$

$\therefore$ The whole time=15 yr. 8 mon. 19 da.

CASE VII.

Given $\left\{\begin{array}{l}\text{Annuity,}\\ \text{Time to Run, and}\\ \text{Present Value of an Annuity,}\end{array}\right\}$ to find the rate of interest.

From the formula (6), Case III, $P=\frac{S}{R-1}\times\frac{R^n-1}{R^n}$, we obtain $\frac{R^n-1}{rR^n}=\frac{P}{S}$ . . . . (1). This is the simplest expression we can obtain for the rate as the equation is of the nth degree and can not be solved in a general manner.

I. If an immediate annuity of \$80, running 14 yr., sells for \$650, what is the rate?

By formula,

$$\frac{R^n-1}{rR^n}=\frac{P}{S}=\frac{\$650}{\$80}=8.125, \text{ or}$$

$\frac{1}{r}-\frac{1}{(1+r)^{14}}=8.125$. Solving this equation by the method of Double Position, we find $r=8\%+$.

By another method.

\$650÷\$80=8.125. By referring to a table of present worths of \$1, corresponding to 14 years, we find it to be between 8 and 9%.

PROBLEMS.

1. What is the amount of an annuity of \$1000, forborne 15 years, at $3\frac{1}{2}\%$ compound interest? *Ans.* \$19295.125

2. What will an annuity of \$30 payable semi-annually, amount to, in arrears 3 years at 7% compound interest?
Ans.——

3. What is the present worth of an annuity of \$500 to continue 40 years at 7%? *Ans.*——

4. What is the present worth of an annuity of \$200, for 7 years, at 5%? *Ans.* \$1152.27.

5. A father presents to his daughter, for 8 years, a rental of \$70 per annum, payable yearly, and the reversion for 12 years succeeding to his son. What is the present value of the gift to his son, allowing 4% compound interest? *Ans.*——

6. A yearly pension which has been forborne for 6 years, at 6%, amounts to \$279; what was the pension? *Ans.* \$480.03.

7. A perpetual annuity of \$100 a year is sold for \$2000; at what rate is the interest reckoned? *Ans.*——

8. A perpetual annuity of \$1000 beginning at the end of 10 years, is to be purchased. If interest is reckoned at $3\frac{1}{2}\%$, what should be paid for it? *Ans.*——

9. If a clergyman's salary of \$700 per annum is 6 years in arrears, how much is due, allowing compound interest at 6%?
Ans. \$4882.72.

10. A soldier's pension of \$350 per annum is 5 years in arrears; allowing 5% compound interest, what is due him?
Ans. \$1933.97.

11. What annual payment will meet principal and interest of a debt of \$2000 due in 4 year a 8% compound interest? *Ans.*—

12. What is the present worth of a perpetual annuity of \$600 at 6% per annum? *Ans.* \$10000.

13. What is the present value of an annuity of \$1000, to commence at the end of 15 years, and continue forever, at 6% per annum? *Ans.* \$6954.40.

14. To what sum will an annuity of \$120 for 20 years amount at 6% per annum? *Ans.* \$4414.27.

15. A debt of \$8000 at 6% compound interest, is discharged by eight equal annual installments; what was the annual installment? *Ans.* \$1288.286.

CHAPTER XV.

MISCELLANEOUS PROBLEMS,

INVOLVING THE VARIOUS APPLICATIONS OF PERCENTAGE.

I. Sold a cow for \$25, losing $16\frac{2}{3}\%$; bought another and sold it at a gain of 16%; I neither gained nor lost on the two; what was the cost of each?

II. A.
1. 100%=cost of the first cow.
2. $16\frac{2}{3}\%$=loss.
3. $100\%-16\frac{2}{3}\%=83\frac{1}{3}\%$=selling price.
4. \$25=selling price.
5. $\therefore 83\frac{1}{3}\%=\$25$,
6. $1\%=\dfrac{1}{83\frac{1}{3}}$ of $\$25=\$.30$, and
7. $100\%=100$ times $\$.30=\$30$=cost of first cow.
8. $\$30-\$25=\$5$, loss on the first cow, and gain on second cow.

B.
1. 100%=cost of second cow.
2. 16%=gain.
3. \$5=gain.
4. $\therefore 16\%=\$5$.
5. $1\%=\frac{1}{16}$ of $\$5=\$.3125$, and
6. $100\%=100$ times $\$.3125=\$31.25$=cost of second cow.

III. $\therefore$ \$30=cost of first cow, and \$31.25=cost of second cow.

Remark.—Since I lost \$5 on the first cow, and neither gained nor lost on the two, I must have gained \$5 on the second cow. $\therefore 16\%=\$5$.

I. There have been two equal annual payments on a 6% note of \$175, given 2 years ago this day. The balance is \$154.40; what was each payment?

II.
(1.) 100%=a payment.
(2.) $100\%=\$175$,
(3.) $1\%=\frac{1}{100}$ of $\$175=\$1.75$, and
(4.) $6\%=6$ times $\$1.75=\$10.50$=interest for 1 year.
(5.) $\$175+\$10.50=\$185.50$=amount before paying the payment.
(6.) $\$185.50-100\%$=amount left after paying the payment.
(7.)
1. $100\%=\$185.50-100\%$,
2. $1\%=\frac{1}{100}$ of $(\$185.50-100\%)=\$1.855-1\%$, and
3. $6\%=6$ times $(\$1.855-6\%)=\$11.13-6\%$=interest for second year.
4. $\$185.50-100\%+\$11.13-6\%=\$196.63-106\%$= amount before paying the last payment.
5. $\$196.63-106\%-100\%=\$196.63-206\%$= amount left after paying the last payment.

(8.) $154.40=amount after paying the last payment
(9.) $\therefore$ $154.40=$196.63—206%.
(10.) 206%=$196.63—$154.40=$42.23,
(11.) 1%=$\frac{1}{206}$ of $42.23=$.205, and
(12.) 100%=100 times $.205=$20.50=the payment.

III. $\therefore$ $20.50=the payment.

Remark.—In this solution we are obliged to use the minus sign, —, which is no obstacle to the student of algebra, but to the student of arithmetic it may seem insurmountable. To avoid this sign, we give another solution.

II.
(1.) 100%=the payment. Then
(2.) $154.40+100%=amount of the debt at the end of the second year.
(3.) 100%=principal that produced this amount.
(4.) 6%=interest.
(5.) 106%=amount.
(6.) $\therefore$ 106%=$154.40+100%, [and
(7.) 1%=$\frac{1}{106}$ of ($154.40+100%)=$1.4566$\frac{2}{53}$+$\frac{50}{53}$%,
(8.) 100%=100 times (1.4566\frac{2}{53}$+$\frac{50}{53}$%) = 145.66\frac{2}{53}$+94$\frac{18}{53}$%=amount at end of the first year after paying off the payment.
(9.) 145.66\frac{2}{53}$+94$\frac{18}{53}$%+100%= 145.66\frac{2}{53}$ + 194$\frac{18}{53}$% =amount before paying off the payment= amount at end of first year.
(10.)
1. 100%=the principal that produced it.
2. 6%=interest.
3. 106%=amount.
4. $\therefore$ 106%=145.66\frac{2}{53}$+194$\frac{18}{53}$%,
5. 1%=$\frac{1}{106}$ of (145.66\frac{2}{53}$+194$\frac{18}{53}$%)=$1.37$\frac{1167}{2809}$ + 1.83$\frac{953}{2809}$%, and
6. 100% = 100 times (1.37\frac{1167}{2809}$ + 1.83$\frac{953}{2809}$%) = 137\frac{1167}{2809}$+183$\frac{953}{2809}$%=the amount at first.
(11.) $175=the amount at first.
(12.) $\therefore$ 137\frac{1167}{2809}$+183$\frac{953}{2809}$%=$175.
(13.) 183$\frac{953}{2809}$%=37\frac{1742}{2809}$.
(14.) 1%=37\frac{1742}{2809}$÷183$\frac{953}{2809}$=$.205, and
(15.) 100%=100 times $.205=$20.50=the payment.

III. $\therefore$ $20.50=the payment. (*R. H. A., p. 264, prob. 5.*)

Explanation.—$154.40=the amount after paying off the last payment. $\therefore$ $154.40+100%=amount before paying of the last payment, or it equals the debt at the end of the first year plus the interest on this debt for the second year. $\therefore$ We let 100%= the debt at the end of the first year, 106%=amount of 100% for 1 year. $\therefore$ 106%=$154.40+100%. Then proceed as in the solution.

I. If a merchant sells $\frac{3}{4}$ of an article for what $\frac{7}{8}$ of it cost, what is his gain %?

II.
1. 100%=cost of whole article.
2. $87\frac{1}{2}\%=\frac{7}{8}$ of 100%=cost of $\frac{7}{8}$ of the article.
3. $87\frac{1}{2}\%$=selling price of $\frac{3}{4}$ of the article.
4. $29\frac{1}{6}\%=\frac{1}{3}$ of $87\frac{1}{2}\%$=selling price of $\frac{1}{4}$ of the article.
5. $116\frac{2}{3}\%$=4 times $29\frac{1}{6}\%$ = selling price of the whole article.
6. $\therefore 116\frac{2}{3}\%-100\%=16\frac{2}{3}\%$=gain.

III. $\therefore 16\frac{2}{3}\%$=his gain. (*Milne's Prac., p. 360, prob. 51.*)

I. A merchant sold goods to a certain amount, on a commission of 4%, and having remitted the net proceeds to the owner, received $\frac{1}{4}\%$ for prompt payment, which amounted to \$15.60. What was his commission?

II.
(1.) 100%=cost of goods.
(2.) 4%=commission.
(3.) $100\%-4\%=96\%$=net proceeds.
(4.)
1. $\frac{1}{4}\%$=amount received for prompt payment.
2. \$15.60=amount received for prompt payment.
3. $\therefore \frac{1}{4}\%$=\$15.60.
4. 1%=4 times \$15.60=\$62.40.
5. 100%=100 times \$62.40=\$6240=net proceeds.

(5.) $\therefore 96\%$=\$6240.
(6.) $1\%=\frac{1}{96}$ of \$6240=\$65, and
(7.) 100%=100 times \$65=\$6500=cost of goods.
(8.)
1. 100%=\$6500.
2. $1\%=\frac{1}{100}$ of \$6500=\$65, and
3. 4%=4 times \$65=\$260=his commission.

III. $\therefore$ His commission=\$260.
(*Greenleaf's N. A., p. 441, prob. 11.*)

I. If I sell 30 yards of cloth for \$132, and gain 10%, how ought I to sell it a yard to lose 25%?

II.
(1.) \$132=selling price of 30 yards.
(2.) \$4.40=\$132÷30=selling price of one yard.
(3.) 100%=cost of one yard.
(4.) 10%=gain.
(5.) $100\%+10\%=110\%$=selling price per yard.
(6.) \$4.40=selling price per yard.
(7.) $\therefore 110\%$=\$4.40.
(8.) $1\%=\frac{1}{110}$ of \$4.40=\$.04,
(9.) 100%=100 times \$.04=\$4=cost per yard.
(10.)
1. 100%=\$4.
2. $1\%=\frac{1}{100}$ of \$4=\$.04,
3. 25%=25 times \$.04=\$1=loss.
4. $\therefore$ \$4—\$1=\$3=selling price per yard to lose 25%.

III. $\therefore$ I must sell it at \$3 per yard to lose 25%.
(*Stoddard's Complete, p. 206, prob. 9.*)

I. A merchant receives on commission three kinds of flour; from A he receives 20 barrels, from B 25 barrels, and from C 40 barrels. He finds that A's flour is 10% better than B's, and that B's is 20% better than C's. He sells the whole at $6 per barrel. What in justice should each man receive?

II.
(1.) $6=selling price of 1 barrel.
(2.) $510=selling price of (20+25+40), or 85 barrels.
(3.) 100%=value of C's flour per barrel.
(4.) 120%=value of B's flour per barrel.
(5.) 1. 100%=120%.
2. 1%=$\frac{1}{100}$ of 120%=$1\frac{1}{5}$%,
3. 10%=10 times $1\frac{1}{5}$%=12%.
(6.) 120%+12%=132%=value of A's flour per barrel.
(7.) 4000%=40 times 100%=what C received.
(8.) 3000%=25 times 120%=what B received.
(9.) 2640%=20 times 132%=what A received.
(10.) 9640%=4000%+3000%+2640%=what all rec'd.
(11.) $510=what all received.
(12.) ∴ 9640%=$510.
(13.) 1%=$\frac{1}{9640}$ of $510=$.52$\frac{211}{241}$, and
(14.) 4000%=4000 times $.52$\frac{211}{241}$=$211$\frac{149}{241}$=what C received.
(15.) 3000%=3000 times $.52$\frac{211}{241}$=$158$\frac{172}{241}$=what B received.
(16.) 2640%=2640 times $.52$\frac{211}{241}$=$139$\frac{161}{241}$=what A received.

III. ∴ 139\frac{161}{241}$=A's share,
158\frac{172}{241}$=B's share, and
211\frac{149}{241}$=C's share.

(*Greenleaf's National Arith. p. 442.*)

I. $\frac{3}{4}$ of B's money equals A's money. What % is A's money less than B's, and what % is B's money more than A's?

II.
1. 100%=B's money.
2. 75%=$\frac{3}{4}$ of 100%=A's money.
3. 100%−75%=25%=excess of B's money over A's.
4. 75%=100% of itself,
5. 1%=$\frac{1}{75}$ of 100%=$1\frac{1}{3}$%, and
6. 25%=25 times $1\frac{1}{3}$%=$33\frac{1}{3}$%=the % B's money is more than A's.

III. ∴ A's money is 25% less than B's, and
B's money is $33\frac{1}{3}$% more than A's money.

(*Stod. Comp., p. 203, prob. 19.*)

I. At what price must an article which cost 30 cents be marked, to allow a discount of $12\frac{1}{2}$% and yield a net profit of $16\frac{2}{3}$%?

II.
- (1.) $100\%=30¢$,
- (2.) $1\%=\frac{1}{100}$ of $30¢=\frac{3}{10}¢$, and
- (3.) $16\frac{2}{3}\%=16\frac{2}{3}$ times $\frac{3}{10}¢=5¢$=profit.
- (4.) $30¢+5¢=35¢$=the price at which it must sell to gain $16\frac{2}{3}\%$.
- (5.)
  1. 100%=marked price.
  2. $12\frac{1}{2}\%$=discount from marked price.
  3. $100\%-12\frac{1}{2}\%=87\frac{1}{2}\%$=selling price.
  4. $35¢$=selling price.
  5. $\therefore 87\frac{1}{2}\%=35¢$.
  6. $1\%=\dfrac{1}{87\frac{1}{2}}$ of $35¢=.40¢$, and
  7. $100\%=100$ times $.40¢=40¢$=marked price.

III. $\therefore 40¢$=marked price.

(*Seymour's Prac., p. 203, prob. 4.*)

I. Had an article cost 10% less, the number of % gain would have been 15% more; what was the gain?

II.
1. *100%*=selling price.
2. 100%=actual cost price.
3. *100%*—100%=gain.
4. 100%—10%=90%=supposed cost.
5. *100%*—90%=conditional gain.
6. 90%=100% of itself.
7. $1\%=\frac{1}{90}$ of $100\%=1\frac{1}{9}\%$.
8. *100%*—90%=(*100*—90) times $1\frac{1}{9}\%=\frac{10}{9}\times$*100%*—100% =conditional gain %. [difference.
9. $\therefore \frac{10}{9}\times$*100%*—100%— (*100%* — 100%) = $\frac{1}{9}\times$*100%* =
10. 15%=difference.
11. $\therefore \frac{1}{9}\times$*100%*=15%. [the actual cost.
12. *100%*=9 times 15%=135%=selling price in terms of
13. $\therefore$ 135%—100%=35%=gain.

III. $\therefore$ 35%=gain. (*R. H. A., p. 406, prob. 87.*)

A literal solution.

Let S=selling price and C=the cost. Then $S-C$=gain and $\dfrac{S-C}{C}$=rate of gain. $S-\frac{9}{10}C$=conditional gain and $\dfrac{S-\frac{9}{10}C}{\frac{9}{10}C}=\dfrac{\frac{10}{9}S-C}{C}$=conditional rate of gain. $\therefore \dfrac{\frac{10}{9}S-C}{C}-\dfrac{S-C}{C}=\frac{3}{20}$, or $\frac{1}{9}S=\frac{3}{20}C$, whence $S=\frac{27}{20}C=1.35C$. $\therefore 1.35C-C=.35C$=gain. $\therefore$ Rate of gain$=.35C\div C=.35=35\%$.

I. In the erection of my house I paid three times as much for material as for labor. Had I paid 6% more for labor, and 10% more for material, my house would have cost \$3052. What did it cost me?

II.
(1.) 100%=cost of labor.
(2.) 300%=3 times 100%=cost of material.
(3.)
1. 100%=100%,
2. 1%=1%, and
3. 6%=6%.
4. 100%+6%=106%=supposed cost of labor.

(4.)
1. 100%=300%,
2. 1%=$\frac{1}{100}$ of 300%=3%, and
3. 10%=10 times 3%=30%.
4. 300%+30%=330%=supposed cost of material.

(5.) 330%+106%=436%=supposed cost of house.
(6.) \$3052=supposed cost of house.
(7.) ∴ 436%=\$3052,
(8.) 1%=$\frac{1}{436}$ of \$3052=\$7, and
(9.) 100%=100 times \$7=\$700=cost of labor.
(10.) 300%=300 times \$7=\$2100=cost of material.
(11.) \$2100+\$700=\$2800=cost of house.

III. ∴ \$2800=cost of the house.

I. I invest $\frac{2}{3}$ as much in 8% canal stock at 104%, as in 6% gas stock at 117%; if my income from both is \$1200, how much did I pay for each, and what was the income from each?

II.
(1.) 100%=investment in gas stock. Then
(2.) $66\frac{2}{3}$%=investment in canal stock.
(3.)
1. 100%=par value of the gas stock.
2. 117%=market value of the gas stock.
3. ∴ 117%=100%, from (1),
4. 1%=$\frac{1}{117}$ of 100%=$\frac{100}{117}$%, and
5. 100%=100 times $\frac{100}{117}$%=$85\frac{25}{39}$%=par value in terms of the investment.

(5.)
1. 100%=$85\frac{25}{39}$%,
2. 1%=$\frac{100}{117}$%, and
3. 6%=6 times $\frac{100}{117}$%=$5\frac{5}{39}$%=income of gas stock.

(6.)
1. 100%=par value of canal stock.
2. 104%=market value.
3. ∴ 104%=$66\frac{2}{3}$%,
4. 1%=$\frac{1}{104}$ of $66\frac{2}{3}$%=$\frac{25}{39}$%, and
5. 100%=100 times $\frac{25}{39}$%=$64\frac{4}{39}$%.

(6.)
1. 100%=$64\frac{4}{39}$%,
2. 1%=$\frac{1}{100}$ of $64\frac{4}{39}$%=$\frac{25}{39}$%, and
3. 8%=8 times $\frac{25}{39}$%=$5\frac{5}{39}$%=income of canal stock.

(7.) $5\frac{5}{39}$%+$5\frac{5}{39}$=$10\frac{10}{39}$%=income from both.
(8.) \$1200=income from both.
(9.) ∴ $10\frac{10}{39}$%=\$1200,
(10.) 1%=$\frac{1}{10\frac{10}{39}}$ of \$1200=\$117, and

(11.) 100%=100 times \$117=\$11700=investment in gas stock.

(12.) $66\frac{2}{3}\%=66\frac{2}{3}$ times \$117=\$7800=investment in canal stock.

(13.) $5\frac{5}{39}\%=5\frac{5}{39}$ times \$117=\$600=income from each.

III. ∴ \$600=income from each.
\$11700=investment in gas stock, and
\$7800=investment in canal stock.

I. A man bought two horses for \$300; he sold them for \$250 apiece. The gain on one was 5% more than on the other; what was the gain on each?

II.
1. \$300=cost of both.
2. \$500=\$250+\$250=selling price of both.
3. \$500—\$300=\$200=gain on both.
4. 100%=gain on first horse. Then
5. 105%=gain on second horse.
6. 205%=100%+105%=gain on both.
7. \$200=gain on both.
8. ∴ 205%=\$200.
9. $1\%=\frac{1}{205}$ of $\$200=\$\frac{40}{41}$, and
10. 100%=100 times $\$\frac{40}{41}=\$97.56\frac{4}{41}$=gain on the first.
11. 105%=105 times $\$\frac{40}{41}=\$102.43\frac{37}{41}$=gain on the second.

III. ∴ $\$97.56\frac{4}{41}$=gain on the first, and
$\$102.43\frac{37}{41}$=gain on the second.

Note.—In this solution, it is assumed that the gain on one was 5% of the gain on the other more than the other, and this is the usual assumption. But the problem really means that the per cent. of gain on one, computed on its cost, was 5% more than the per cent. of gain on the other, computed on its cost. By this assumption, the problem is algebraic. The following is the solution: Let x=the cost of the first horse, and \$300—x, the cost of the second. Then \$250—x=gain on first, and \$250—(\$300—x)=x—\$50, the gain on the second. (\$250—x)÷x= rate of gain on the first, and (x—\$50)÷(\$300—x), the rate of gain on the second. Then $(250-x)\div x-(x-50)\div(300-x)=\frac{1}{20}$. Whence, by clearing of fractions, transposing and, combining, $x^2-10300x=-1500000$, $x=5150\pm50\sqrt{10009}=\$147.7755$, the cost of the first horse. \$300—x=\$152.2245, the cost of the second horse. \$250—x=\$102.2245, gain on the first horse, and x—\$50=\$97.7755, the gain on the second horse.

I. An agent sells produce at 2% commission, invests the proceeds in flour at 3% commission; his whole commission was \$75. How many barrels of flour did he buy at \$5 per barrel?

II.
(1.) 100%=value of the produce.
(2.) 2%=the commission.
(3.) $100\%-2\%=98\%$= net proceeds, or amount invested in the flour.
(4.)
1. 100%=cost of the flour.
2. 3%=commission on flour.
3. $100\%+3\%=103\%$=whole cost of the flour.
4. $\therefore 103\%=98\%$,
5. $1\%=\frac{1}{103}$ of $98\%=\frac{98}{103}\%$, and
6. $100\%=100\times\frac{98}{103}\%=95\frac{15}{103}\%$ = cost of flour in terms of the value of the produce.
7. $98\%-95\frac{15}{103}\%=2\frac{88}{103}\%$=commission on flour.
(5.) $2\%+2\frac{88}{103}\%=4\frac{88}{103}\%$=whole commission.
(6.) \$75=whole commission.
(7.) $\therefore 4\frac{88}{103}\%=\$75$,
(8.) $1\%=\$75\div4\frac{88}{103}=\$15.45$, and
(9.) $100\%=100$ times $\$15.45=\$1545$ = value of the produce.
(10.) $95\frac{15}{103}\%=95\frac{15}{103}$ times $\$15.45=\$1470$ = value of the the flour.
(11.) \$5=cost of 1 barrel.
(12.) \$1470=cost of $1470\div5$, or 294 barrels.

III. $\therefore$ The agent bought 294 barrels of flour.

I. A distiller sold his whisky, losing 4%; keeping \$18 of the proceeds, he gave the remainder to an agent to buy rye at 8% commission; he lost in all \$32; what was the whisky worth?

II
(1.) 100%=value of the whisky.
(2.) 4%=loss.
(3.) $100\%-4\%=96\%$=amount he had left.
(4.) $96\%-\$18$=amount he invested in rye.
(5.)
1. 100%=cost of the rye.
2. 8%=commission on the rye.
3. $100\%+8\%=108\%$=whole cost of rye.
4. $\therefore 108\%=96\%-\$18$.
5. $1\%=\frac{1}{108}$ of $(96\%-\$18)=\frac{8}{9}\%-\$.16\frac{2}{3}$, and
6. $100\%=100$ times $(\frac{8}{9}\%-\$.16\frac{2}{3})=88\frac{8}{9}\%-\$16.66\frac{2}{3}$ =cost of rye.
7. $8\%=8$ times $(\frac{8}{9}\%-\$.16\frac{2}{3})=7\frac{1}{9}\%-\$1.33\frac{1}{3}$=commission on rye.
(6.) $4\%+(7\frac{1}{9}\%-\$1.33\frac{1}{3})=11\frac{1}{9}\%-\$1.33\frac{1}{3}$=whole loss.
(7.) \$32=whole loss.
(8.) $\therefore 11\frac{1}{9}\%-\$1.33\frac{1}{3}=\$32$
(9.) $11\frac{1}{9}\%=\$33.33\frac{1}{3}$,
(10.) $1\%=\frac{1}{11\frac{1}{9}}$ of $\$33.33\frac{1}{3}=\$3$, and
(11.) $100\%=100$ times $\$3=\$300$=value of the whisky.

III. $\therefore$ \$300=value of the whisky.

(*R. H. A., p. 406, prob 91.*)

I. What will be the cost in New Orleans of a draft on New York, payable 60 days after sight, for \$5000, exchange being at $1\frac{1}{2}\%$ premium?

II.
1. 100%=face of the draft.
2. $1\frac{1}{2}\%$=premium.
3. $100\%+1\frac{1}{2}\%=101\frac{1}{2}\%$=rate of exchange.
4. 5%=discount for one year.
5. $\frac{7}{8}\%=\frac{63}{360}$ of 5%=discount for 63 days.
6. $\therefore$ $101\frac{1}{2}\%-\frac{7}{8}\%=100\frac{5}{8}\%$=cost of the draft
7. 100%=\$5000.
8. $1\%=\frac{1}{100}$ of \$5000=\$50, and
9. $100\frac{5}{8}\%=100\frac{5}{8}$ times \$50=\$5031.25=cost of the draft.

III. $\therefore$ \$5031.25=cost of the draft.

Explanation.—Observe that since the draft is not to be paid in New York for 63 days, the banker in New Orleans, who has the use of the money for that time allows the drawer discount on the face of the draft for that time, which goes (1) towards reducing the premium if there be any, and (2) towards reducing the face of the draft.

Note.—The rate of exchange between two places or countries depends upon the course of trade. Suppose the trade between New York and New Orleans is such that New York owes New Orleans \$10,250,000 and New Orleans owes New York \$13,000,000. There is a "balance of trade" of \$2,750,000 against New Orleans and in favor of New York. Hence, the demand in New Orleans for drafts on New York is greater than the demand in New York for drafts on New Orleans and, therefore, the drafts are at a premium in New Orleans. But if New York owes New Orleans \$13,000,000 and New Orleans owes New York \$10,250,000, the "balance of trade," \$2,750,000, is *against* New York and in *favor* of New Orleans. Hence, the demand in New Orleans for drafts on New York is less than the demand in New York for drafts on New Orleans and, therefore, the drafts are at a discount in New Orleans.

If the trade between the two places is the same, the rate of exchange is at par.

The reason why the banks in New York should charge a premium, when the balance of trade is against them, is that they must be at the expense of actually sending money to the New Orleans banks or be charged interest on their unpaid balance; the reason why the New Orleans banks will sell at a discount is that they are willing to sell for less than the face of a draft in order to get the money owed them in New York immediately.

Exchange is charged from $\frac{1}{8}$ to $\frac{1}{2}\%$, and is designed to cover the cost of transporting the funds from one place to another.

I. What will a 30 days' draft on New Orleans for \$7216.85 cost, at $\frac{5}{8}\%$ discount, interest 6%?

II.
1. 100%=face of draft.
2. $\frac{5}{8}\%$=discount.
3. $100\%-\frac{5}{8}\%=99\frac{3}{8}\%$=face less the discount.
4. 6%=bank discount for 1 year.
5. $\frac{11}{20}\%=\frac{33}{360}$ of 6%=bank discount for 33 days.
6. $99\frac{3}{8}\%-\frac{11}{20}\%=99\frac{33}{40}\%$=cost of the draft.
7. 100%=\$7216.85,
8. $1\%=\frac{1}{100}$ of \$7216.85=\$72.1685, and
9. $99\frac{33}{40}\%=99\frac{33}{40}$ times \$72.1685=\$7150.094=cost of the draft.

III. ∴ \$7150.094=cost of the draft.

I. The aggregate face value of two notes is \$761.70 and each has 1 year 3 months to run; one of the notes I had discounted at 10% true discount and the other at 10% bank discount, and realized from both notes \$671.50. Find the face value of both notes.

II.
(1.) 100%=face of note discounted at bank discount.
(2.) \$761.70—100%=face of note discounted at true discount.
(3.) 10%=bank discount for 1 year.
(4.) $12\frac{1}{2}\%$=bank discount for 1 year 3 months.
(5.)
1. 100%=present worth of second note.
2. 10%=interest on present worth for 1 year.
3. $12\frac{1}{2}\%$=interest for 1 year 3 months.
4. $100\%+12\frac{1}{2}\%=112\frac{1}{2}\%$=amount of present worth.
5. \$761.70—100%=amount of the present worth.
6. ∴ $112\frac{1}{2}\%$=\$761.70—100%,
7. $1\%=\frac{1}{112\frac{1}{2}}$ of (\$761.70—100%)=\$6.7706$\frac{2}{3}$—$\frac{8}{9}\%$,
8. 100%=100 times (\$6.7706$\frac{2}{3}$ — $\frac{8}{9}\%$) = \$677.06$\frac{2}{3}$—$88\frac{8}{9}\%$=present worth.

(6.) \$761.70 — 100%—(\$677.06$\frac{2}{3}$—$88\frac{8}{9}\%$) = \$84.63$\frac{1}{3}$ — $11\frac{1}{9}\%$=true discount.
(7.) \$84.63$\frac{1}{3}$—$11\frac{1}{9}\%+12\frac{1}{2}\%$=\$84.63$\frac{1}{3}$+ $1\frac{7}{18}\%$ = whole [discount.
(8.) \$761.70—\$671.50=\$90.20=whole discount.
(9.) ∴\$84.63$\frac{1}{3}$+$1\frac{7}{18}\%$=\$90.20,
(10.) $1\frac{7}{18}\%$=\$5.56$\frac{2}{3}$,
(11.) $1\%=\frac{1}{1\frac{7}{18}}$ of \$5.56$\frac{2}{3}$=\$4.008, and
(12.) 100%=100 times \$4.008=\$400.80=face of note discounted at bank discount.
(13.) \$761.70—100%=\$761.70—\$400.80=\$360.90=face of note discounted at true discount.

III. ∴ { \$400.80=face of note discounted at bank discount, and
\$360.90=face of note discounted at true discount.

1. A merchant sold part of his goods at a profit of 20%, and the remainder at a loss of 11%. His goods cost \$1000 and his gain was \$100; how mnch was sold at a profit?

II.
(1.) 100%=cost of goods sold at a profit. Then
(2.) \$1000—100%=cost of goods sold at a loss.
(3.) 20%=profit on 100%, the part sold at a profit.
(4.)
1. 100%=\$1000—100%.
2. 1%=$\frac{1}{100}$ of (\$1000—100%)=\$10—1%,
3. 11%=11 times (\$10—1%)=\$110—11%=loss on the remainder.
(5.) ∴ 20%—(\$110—11%)=31%—\$110=gain.
(6.) \$100=gain.
(7.) ∴ 31%—\$110=\$100.
(8.) 31%=\$210,
(9.) 1%=$\frac{1}{31}$ of \$210=\$6$\frac{24}{31}$,
(10.) 100%=100 times \$6$\frac{24}{31}$=677.41$\frac{29}{31}$=part sold at a [profit.

III. ∴ \$677.41$\frac{29}{31}$=value of the part sold at a profit.

I. By discounting a note at 20% per annum, I get 22$\frac{1}{2}$% per annum interest; how long does the note?

II.
1. 22$\frac{1}{2}$% of the proceeds=20% of the face of the note.
2. 1% of the proceeds=$\frac{1}{22\frac{1}{2}}$ of 20%=$\frac{8}{9}$% of the face of the note.
3. 100% of the proceeds=100 timss $\frac{8}{9}$%=88$\frac{8}{9}$% of the face of the note.
4. 100%=face of the note.
5. 88$\frac{8}{9}$%=proceeds.
6. 100%—88$\frac{8}{9}$%=11$\frac{1}{9}$%=discount for a certain time.
7. 20%=discount for 360 days.
8. 1%=discount for $\frac{1}{20}$ of 360 days, or 18 days.
9. 11$\frac{1}{9}$%=discount for 11$\frac{1}{9}$ times 18 days, or 200 days.

III. ∴ The note was discounted for 200 days.

I. A man bought a farm for \$5000, agreeing to pay principal and interest in 5 equal annual installments. What will be the annual payment including interest at 6%?

(1.)
1. 100%=one annual payment.
2. ∴ 100%=amount paid at end of the fifth year since the debt was then discharged.
3. 100%=principal that drew interest the fifth year.
4. 6%=interest on this principal.
5. ∴ 100%+6%=106%=amount of this principal.
6. ∴ 106%=100%=the annual payment.
7. 1%=$\frac{1}{106}$ of 100%=$\frac{50}{53}$%, and
8. 100%=100 times $\frac{50}{53}$%=94$\frac{18}{53}$%=principal at the beginning of the fifth year.
9. 94$\frac{18}{53}$%+100%=194$\frac{18}{53}$%=amount at the end of the fourth year.

II.

(2.)
1. 100%=principal at the beginning of the fourth year.
2. 6%=interest on this principal.
3. $100\%+6\%=106\%$=amount.
4. $\therefore 106\%=194\frac{18}{53}\%$,
5. $1\%=\frac{1}{106}$ of $194\frac{18}{53}\%=1.83\frac{953}{2809}\%$, and
6. $100\%=100$ times $1.83\frac{953}{2809}\%=183\frac{953}{2809}\%$=principal at the beginning of the fourth year.
7. $183\frac{953}{2809}\%+100\%=283\frac{953}{2809}\%$=amount at the end of the third year.

(3.)
1. 100%=principal at the beginning of the third year.
2. 6%=interest.
3. $100\%+6\%=106\%$ = amount at the end of the third year.
4. $\therefore 106\%=283\frac{953}{2809}\%$,
5. $1\%=\frac{1}{106}$ of $283\frac{953}{2809}\%=2.67\frac{44841}{148877}\%$, and
6. $100\%=100$ times $2.67\frac{44841}{148877}\% = 267\frac{44841}{148877}\%$ = principal at the beginning of third year.
7. $267\frac{44841}{148877}\%+100\% = 367\frac{44841}{148877}\%$ = amount at the end of second year.

(4.)
1. 100%=principal at the beginning of second year.
2. 6%=interest.
3. $100\%+6\%=106\%$=amount at the end of second year.
4. $\therefore 106\%=367\frac{44841}{148877}\%$,
5. $1\%=\frac{1}{106}$ of $367\frac{44841}{148877}\%=3.46\frac{4028574}{7890481}\%$, and
6. $100\%=100$ times $3.46\frac{4028574}{7890481}\%=346\frac{4028574}{7890481}\%$= principal at the beginning of the second year.
7. $346\frac{4028574}{7890481}\%+100\%=446\frac{4028574}{7890481}\%$=amount at the end of first year.

(5.)
1. 100%=principal at the beginning of the first year, or the cost of farm.
2. 6%=interest.
3. $100\%+6\%=106\%$=amount at end of first year.
4. $\therefore 106\%=446\frac{4028574}{7890481}\%$,
5. $1\%=\frac{1}{106}$ of $446\frac{4028574}{7890481}\%=4.21\frac{98852447}{418195493}\%$, and
6. $100\%=100$ times $4.21\frac{98852447}{418195493}\%=421\frac{98852447}{418195493}\%$=cost of the farm.

(6.) \$5000=cost of the farm.

(7.) $\therefore 421\frac{98852447}{418195493}\%$=\$5000,

(8.) 1%=\$5000$\div 421\frac{98852447}{418195493}$=\$11.8698+, and

(9.) $100\%=100$ times \$11.8698=\$1186.98+=the annual payment.

III. $\therefore$ \$1186.98+=the annual payment.

(*Milne's Prac., p. 361, prob. 63.*)

I. A and B have \$4700; $\dfrac{\frac{3\%}{4}\%}{\frac{1\%}{2}\%}\%$ of A's share equals $\dfrac{\frac{2}{3\%}\%}{60\%}\%$ of B's share; how much has each?

II.

1. $\frac{3\%}{4}\%=\frac{\frac{3}{100}}{4}\%=\frac{3}{400}\%=\frac{3}{40000}$.
2. $\frac{1\%}{2}\%=\frac{\frac{1}{100}}{2}\%=\frac{1}{200}\%=\frac{1}{20000}$.
3. $\therefore \frac{\frac{3\%}{4}\%}{\frac{1\%}{2}\%}\%=\frac{\frac{3}{40000}}{\frac{1}{20000}}\%=1\frac{1}{2}\%$.
4. $\frac{2}{3\%}\%=\frac{2}{\frac{3}{100}}\%=\frac{200}{3}\%=\frac{200}{300}=\frac{2}{3}$.
5. $60\%=\frac{60}{100}=\frac{3}{5}$.
6. $\therefore \frac{\frac{2}{3\%}\%}{60\%}\%=\frac{\frac{2}{3}}{\frac{3}{5}}\%=1\frac{1}{9}\%$.
7. $\therefore 1\frac{1}{2}\%$ of A's$=1\frac{1}{9}\%$ of B's,
8. 1% of A's$=\frac{1}{1\frac{1}{2}}$ of $1\frac{1}{9}\%=\frac{20}{27}\%$ of B's, and
9. 100% of A's$=100$ times $\frac{20}{27}\%=74\frac{2}{27}\%$ of B's.
10. 100%=B's share.
11. $74\frac{2}{27}\%$=A's share.
12. $100\%+74\frac{2}{27}\%=174\frac{2}{27}\%$=sum of their shares.
13. $4700=sum of their shares.
14. $\therefore 174\frac{2}{27}\%=\$4700$,
15. $1\%=\frac{1}{174\frac{2}{27}}$ of $\$4700=\$27$, and
16. $100\%=100$ times $\$27=\$2700$=B's share.
17. $74\frac{2}{27}\%=74\frac{2}{27}$ times $\$27=\$2000$=A's share.

III. $\therefore$ $\begin{cases}\$2700=\text{B's share and}\\ \$2000=\text{A's share.}\end{cases}$

CHAPTER XVI.

RATIO AND PROPORTION.

1. *Ratio* is the relative magnitude of one quantity as compared with another of the same kind; thus, the ratio of 12 apples to 4 apples is 3.

The first quanity, 12 apples, is called the *Antecedent*, and the second quantity, 4 apples, the consequent. Taken together they are called *Terms* of the ratio, or a *Couplet.*

2. *The Sign* of ratio is the colon, :, the common sign of division with the horizontal line omitted.

Note.—Olney says, "There is a common notion among us, that the French express a ratio by divding the consequent by the antecedent, while the English express it by dividing the antecedent by the consequent. Such is not the fact. French, German, and English writers agree in the above definition. In fact, the Germans very generally use the sign : instead of $\div$; and

by all, the two signs are used as exact equivalents." Some writers, however, divide the consequent by the antecedent, as $a : b = \frac{b}{a}$. This is according to Webster's definition and illustration. To my mind, to divide the antecedent by the consequent is more simple and philosophical and should be universally adopted by all writers on mathematics.

3. ***A Direct Ratio*** is the quotient of the antecedent divided by the consequent.

4. ***An Indirect Ratio*** is the quotient of the consequent by the antecedent.

5. A ratio of ***Greater Inequality*** is a ratio greater than unity; as, 7:3.

6. A ratio of ***Less Inequality*** is a ratio less than unity; as, 4:5.

7. ***A Compound Ratio*** is the product of the corresponding terms of several simple ratios. Thus, the compound ratio of 1:3, 5:4, and 7:2 is $1\times5\times7:3\times4\times2$.

8. ***A Duplicate Ratio*** is the ratio of the squares of two numbers.

9. ***A Triplicate Ratio*** is the ratio of the cubes of two numbers; as, $a^3 : b^3$.

10. ***A Subduplicate Ratio*** is the ratio of the square roots of two numbers; as, $\sqrt{a} : \sqrt{b}$.

11. ***A Subtriplicate Ratio*** is the ratio of the cube roots of two numbers; as, $\sqrt[3]{a} : \sqrt[3]{b}$.

PROPORTION.

12. ***Proportion*** is an equality of ratios. The equality is indicated by the ordinary sign of equality or by the double colon, ::. Thus. $a : b = c : d$, or $a : b :: c : d$.

13. ***The Extremes*** of a proportion are the first and fourth terms.

14. ***The Means*** are the second and third terms.

15. ***A Mean Proportional*** between two quantities is a quantity to which either of the two quantities bears the same ratio that the mean does to the other of the two.

16. ***A Continued Proportion*** is a succession of equal ratios, in which each consequent is the antecedent of the next ratio.

17. ***A Compound Proportion*** is an expression of equality between a compound and a simple ratio.

18. ***A Conjoined Proportion*** is a proportion which has each antecedent of a compound ratio equal in value to its consequent. The first of each pair of equivalent terms is an antecedent, and the term following, a consequent. This is also called the "Chain rule."

I. What is the ratio of $\frac{1}{2}$ to $\frac{2}{3}$?
$\frac{1}{2}\div\frac{2}{3}=\frac{1}{2}\times\frac{3}{2}=\frac{3}{4}$, the ratio.

I. What is the ratio of 10 bu. to $1\frac{3}{7}$ bu.?
10 bu. $\div\ 1\frac{3}{7}$ bu. $=10\times\frac{7}{10}=7$, the ratio.

I. What is the ratio of 25 apples to 75 boxes?
Ans. No ratio; for no number of times one will produce the other

In a true proportion, we must always have greater : less :: greater : less or less : greater :: less : greater. The test for the truth of a proportion is that the product of the means equals the product of the extremes.

I. If a 5-cent loaf weighs 7oz. when flour is \$8 per barrel, how much should it weigh when flour is \$7.50 per barrel?

It should evidently weigh more.
$\therefore$ less : greater :: less : greater.
\$7.50 : \$8.00 :: 7oz : (? $=7\frac{7}{15}$oz.)

I. If a staff 3 feet long, casts a shadow 2 feet, how high is the steeple whose shadow at the same time is 75 feet?

Since the steeple casts a longer shadow than the staff, it is evidently higher than the staff.
$\therefore$ less : greater :: less : greater.
2 feet : 75 feet :: 3 feet : (? $=112\frac{1}{2}$ feet.)

I. What number is that which being divided by one more than itself, gives $\frac{1}{7}$ for a quotient?

II.
1. Let $\frac{2}{2}$=number. Then
2. $\dfrac{\frac{2}{2}}{\frac{2}{2}+1}=\frac{1}{7}$ or $\frac{2}{2}:\frac{2}{2}+1::1:7$, whence
3. $7(\frac{2}{2})=1(\frac{2}{2}+1)$ or
4. $\frac{14}{2}=\frac{2}{2}+1$; whence
5. $\frac{12}{2}=1$,
6. $\frac{1}{2}=\frac{1}{12}$, and
7. $\frac{2}{2}=2$ times $\frac{1}{12}=\frac{1}{6}$=number.

III. $\therefore\ \frac{1}{6}$=the number.

I. What number divided by 3 more than itself gives $\frac{7}{9}$ for a quotient?

II.
1. Let $\frac{2}{2}$=the number. Then
2. $\frac{\frac{2}{2}}{\frac{2}{2}+3}=\frac{7}{9}$ or, putting this in the form of a proportion,
3. $\frac{2}{2} : \frac{2}{2}+3 :: 7 : 9$. [the product of the extremes.
4. ∴ $\frac{18}{2}=\frac{14}{2}+21$, the product of the means being equal to
5. $\frac{18}{2}-\frac{14}{2}=\frac{4}{2}=21$,
6. $\frac{1}{2}=\frac{1}{4}$ of $21=5\frac{1}{4}$, and
7. $\frac{2}{2}=2$ times $5\frac{1}{4}=10\frac{1}{2}$=the number.

III. ∴ $10\frac{1}{2}$=the number.

I. If 7 lb. of coffee is equal in value to 5 lb. of tea, and 3 lb. of tea to 13 lb. of sugar, 39 lb. of sugar to 24 lb. of rice, 12 lb. of rice to 7 lb. of butter, 8 lb. of butter to 12 lb. of cheese; how many lb. of coffee are equal in value to 65 lb. of cheese?

II.
1. 7 lb. of coffee=5 lb. of tea,
2. 3 lb. of tea=13 lb. of sugar,
3. 39 lb. of sugar=24 lb. of rice,
4. 12 lb. of rice=7 lb. of butter,
5. 8 lb. of butter=12 lb. of cheese, and
6. 65 lb. of cheese= ?=39 lb. of coffee,
7. $\frac{7\times3\times39\times12\times8\times65}{5\times13\times24\times7\times12}=39$ lb.

III. ∴ 65 lb. of cheese=39 lb. of coffee.

I. I can keep 10 horses or 15 cows on my farm; how many horses can I keep if I have 9 cows?

15 cows : 9 cows :: 10 horses : ?=6 horses.

10 horses—6 horses=4 horses.

∴ I can keep 4 horses with the 9 cows.

I. If 2 oxen or 3 cows eat one ton of hay in 60 days, how long will it last 4 oxen and 5 cows?

2 oxen : 4 oxen :: 3 cows : ?=6 cows.

∴ 4 oxen eat as much as 6 cows. If a ton of hay last 3 cows 60 days, it will last 6 cows, which are equal to 4 oxen, and 5 cows, or 11 cows, not so long.

∴ 11 cows : 3 cows :: 60 days : ?=$17\frac{3}{11}$ days.

I. If 24 men, by working 8 hours a day, can, in 18 days, dig a ditch 95 rods long, 12 feet wide at the top, 10 feet wide at the bottom, and 9 feet deep; how many men, in 24 days of 12 hours a day, will be required to dig a ditch 380 rods long, 9 feet wide at the top, 5 feet wide at the bottom, and 6 feet deep?

$$\left.\begin{array}{l}95 \text{ rods} : 380 \text{ rods}\\ 24 \text{ days} : 18 \text{ days}\\ 12 \text{ hours} : 8 \text{ hours}\\ 12 \text{ feet} : 9 \text{ feet}\\ 10 \text{ feet} : 5 \text{ feet}\\ 9 \text{ feet} : 6 \text{ feet}\end{array}\right\} :: 24 \text{ men} : ?=12 \text{ men}.$$

$$\frac{380\times18\times8\times9\times5\times6\times24}{95\times24\times12\times12\times10\times9}=12 \text{ men.}$$

I. A Louisville merchant wishes to pay \$10000, which he owes in Berlin. He can buy a bill of exchange in Louisville on Berlin at the rate of \$.96 for 4 reichmarks; or he is offered a circular bill through London and Paris, brokerage $\frac{1}{8}\%$ at each place, at the following rates: £1=\$4.90=25.38 francs, and 5 francs=4 reichmarks. What does he gain by direct exchange?

II.
1. \$.238=1 mark.
2. \$10000=10000÷.238=42016.807 marks.
3. \$.24=1 mark, since this is the rate of exchange.
4. ∴ \$10084.033=42016.807 times \$.24=42016.807 marks =direct exchange.
5. 42016.807 marks=(?=\$10165.38.)
6. \$4.90=£1—$\frac{1}{8}\%$ of £1=£.99$\frac{7}{8}$.
7. £1=.99$\frac{7}{8}$ times 25.38 fr.
8. 5 fr.=4 marks.
9. $\frac{42016.807\times4.90\times5}{.99\frac{7}{8}\times.99\frac{7}{8}\times25.38\times4}$=\$10165.38=cost by circular exchange.
10. \$10165.38—\$10084.033=\$81.35 = gain by direct exchange.

III. ∴ \$81.35=gain by direct exchange.

I. A wheel has 35 cogs; a smaller wheel working in it, 26 cogs; in how many revolutions of the larger wheel will the smaller one gain 10 revolutions?

II.
1. 35 cogs—26 cogs=9 cogs=what the smaller wheel gains on larger in 1 revolution of larger wheel.
2. 26 cogs passed through the point of contact=1 revolution of smaller wheel.
3. 1 cog passed through the point of contact=$\frac{1}{26}$ revolution of smaller wheel.
4. 9 cogs passed through the point of contact=$\frac{9}{26}$ revolution of smaller wheel.
5. ∴ In 1 revolution of larger wheel the smaller gains $\frac{9}{26}$ revolution of smaller wheel.
6. ∴ $\frac{9}{26}$ revolution gained : 10 revolutions gained : : 1 revolution of larger wheel : ?=28$\frac{8}{9}$ revolutions of larger wheel.

III. ∴ The smaller wheel will gain 10 revolutions in 28$\frac{8}{9}$ revolutions of larger wheel.

By analysis and proportion.

26 cogs passed through the point of contact=1 revolution of the smaller wheel.

35 cogs passed through the point of contact=1 revolution of the larger wheel. But when the larger wheel has made 1 revolution, 35 cogs of the smaller wheel have passed through the point of contact. If 26 cogs having passed through the point of contact make 1 revolution of the smaller wheel, how many revolutions will 35 cogs make?

By proportion, 26 cogs : 35 cogs :: 1 rev. : ?=$1\frac{9}{26}$ rev.

∴ The smaller wheel makes $1\frac{9}{26}$ revolutions while the larger wheel makes 1 revolution. ∴ The smaller gains $1\frac{9}{26}$ revolutions —1 revolution=$\frac{9}{26}$ revolution. If the smaller wheel gains $\frac{9}{26}$ revoluion in 1 revolution of the larger wheel to gain 10 revolutions on the larger wheel, the larger wheel must make more revolutions. ∴ less : greater :: less : greater.

$\frac{9}{26}$ rev. : 10 rev. :: 1 rev. of larger : ?=$28\frac{8}{9}$ rev. of larger.

I. If the velocity of sound be 1142 feet per second, and the number of pulsations in a person 70 per minute, what is the distance of a cloud, if 20 pulsations are counted between the time of seeing the flash and hearing the thunder?

II.
1. 1142 ft.=distance sound travels in 1 second.
2. 68520 ft.=60×1142 ft.=distance sound travels in 1 min., or the time of 70 pulsations.
3. ∴ If it travels 68520 feet while 70 pulsations are counted, it will travel not so far while 20 pulsations are counted.
4. ∴ greater : less :: greater : less.
5. 70 pul. : 20 pul. :: 68520 ft. : ?=$19577\frac{1}{7}$ ft.=3 mi. 5 fur. [145 yd. $2\frac{1}{7}$ ft.

III. ∴ The cloud is 3 mi. 5 fur. 145 yd. $2\frac{1}{7}$ ft. distant.

(*R., 3d p., p. 289, prob. 45.*)

PROBLEMS.

1. If 3 horses, in $\frac{1}{4}$ of a month eat $\frac{3}{4}$ of a ton of hay, how long will $\frac{5}{6}$ of a ton last 5 horses?

2. If a 4-cent loaf weighs 9 oz. when flour is \$6 a barrel, how much ought a 5-cent loaf weigh when flour is \$8 per barrel?

3. A dog is chasing a hare, which is 46 rods ahead of the dog. The dog runs 19 rods while the hare runs 17; how far must the dog run before he catches the hare?

4. If 52 men can dig a trench 355 feet long, 60 feet wide, and 8 feet deep in 15 days, how long will a trench be that is 45 feet wide and 10 feet deep, which 45 men can dig in 25 days?

5. If $\frac{1}{6}$ of 12 be 3 what will $\frac{1}{4}$ of 40 be? *Ans.* 15.

6. If 3 be $\frac{1}{6}$ of 12, what will $\frac{1}{4}$ of 40 be? *Ans.* $6\frac{2}{3}$.

7. If 18 men or 20 women do a work in 9 days, in what time can 4 men and 9 women do the same work? *Ans.* $6\frac{1}{20}$ days.

8. If 5 oxen or 7 cows eat $3\frac{4}{11}$ tons of hay in 87 days, in what time will 2 oxen and 3 cows eat the same quantity of hay? *Ans.* 105 days.

9. Divide \$600 between three men, so that the second man shall receive one-third more than the first, and the third $\frac{2}{3}$ more than the second.

10. Two men in Boston hire a carriage for \$25, to go to Concord, N. H., and back, the distance being 72 miles, with the privilege of taking in three more persons. Having gone 20 miles, they took in A; at Concord they took in B; and when within 30 miles of Boston, they took in C. How much shall each pay? *Ans.* First man, \$7.609$\frac{103}{108}$; second, \$7.609$\frac{103}{108}$; A., \$5.873$\frac{91}{108}$; B, \$2,864$\frac{7}{12}$; and C, \$1.041$\frac{8}{12}$.

11. Three men purchased 6150 sheep. The number of A's sheep is to the number of B's sheep as $\frac{2}{3}$ is to $3\frac{1}{3}$, and 4 times the number of C's sheep is to the number of A's sheep as $\frac{1}{3}$ is to $\frac{1}{9}$. Find the number of sheep each had.

Ans. $\begin{cases} \text{A's} = \\ \text{B's} = \\ \text{C's} = \end{cases}$

12. If \$500 gain \$10 in 4 months, what is the rate per cent? *Ans.* 8%.

13. If 12 men can do as much work as 25 women, and 5 women do as much as 6 boys; how many men would it take to do the work of 75 boys? *Ans.* 30 men.

14. If 5 experienced compositors in 16 days, 11 hours each, can compose 25 sheets of 24 pages in each sheet, 44 lines on a page, 8 words in a line, and 5 letters to a word; how many inexperienced compositors in 12 days, 10 hours each, will it take to compose a volume (to be printed with the same kind of type), consisting of 36 sheets, 16 pages to a sheet, 100 lines to the page, 5 words to a line, and 9 letters to a word, provided that while composing an inexperienced compositor can do only $\frac{4}{5}$ as much as an experienced compositor, and that the latter work is only $\frac{5}{8}$ as hard as the former? *Ans.* 16.

15. If A can do $\frac{2}{3}$ as much in a day as B, B can do $\frac{3}{4}$ as much as C, and C can do $\frac{4}{5}$ as much as D, and D can do $\frac{5}{6}$ as much as E, and E can do $\frac{6}{7}$ as much as F; in what time can F do as much work as A can do in 28 days? *Ans.* 8.

16. A starts on a journey, and travels 27 miles a day; 7 days after, B starts, and travels the same road, 36 miles a day; in how many days will B overtake A? *Ans.* 21 days.

17. A wheel has 45 cogs; a smaller wheel working in it, 36 cogs; in how many revolutions of the larger wheel will the smaller gain 10 revolutions? *Ans.* 40.

18. If the velocity of sound be 1142 feet per second, and the number of pulsations in a person 70 per minute, what is the distance of a cloud, if 30 pulsations are counted between the time of seeing a flash of lightning and hearing the thunder?

Ans. $5\frac{1}{2}$ mi. 108 yd. $1\frac{3}{7}$ ft.

19. If William's services are worth $\$15\frac{2}{3}$ a month, when he labors 9 hours a day, what ought he to receive for $4\frac{2}{5}$ months, when he labors 12 hours a day? *Ans* $\$91.91\frac{1}{9}$.

20. If 300 cats kill 300 rats in 300 minutes. how many cats will kill 100 rats in 100 minutes? *Ans.* 300 cats.

CHAPTER XVII.

ANALYSIS.

1. ***Analysis,*** in mathematics, is the process of solving problems by tracing the relation of the parts.

I. What will 7 lb. of sugar cost at 5 cents a pound?

Analysis for primary classes.

If one ponnd of sugar costs 5 cents, 7 pounds will cost 7 times 5 cents, which are 35 cents.

I. If 6 lead pencils cost 30 cents, what will one lead pencil cost?

Analysis: If 6 lead pencils cost 30 cents, one lead pencil will cost as many cents as 6 is contained into 30 cents which are 5 cents.

I. If 8 cranges cost 48 cents, what will 5 oranges cost?

Analysis: If 8 oranges cost 48 cents, one orange will cost as many cents as 8 is contained into 48 cents which are 6 cents; if one orange costs 6 cents 5 oranges will cost 5 times 6 cents, which are 30 cents.

I. If a boy had 7 apples and ate 2 of them, how many had he left?

Analysis: If a boy had 7 apples and ate 2 of them, he had left the difference between 7 apples and 2 apples which are 5 apples.

I. If John had 12 cents and found 5 cents, how many cents did he then have?

Analysis: If John had 12 cents and found 5 cents, he then had the sum of 12 cents and 5 cents which are 17 cents.

Note.—If teachers in the Primary Departments would see that their pupils gave the correct analysis to such problems, their pupils would often be better prepared for the higher grades. After they are thoroughly acquainted with the analysis of such questions they may be taught to write out neat, accurate solutions with far less trouble than if allowed to give careless analysis to problems in the lower grades.

I. If 4 balls cost 36 cents, how many balls can be bought for 81 cents?

Analysis: If 4 balls cost 36 cents, one ball will cost as many cents as 4 is contained into 36 cents which are 9 cents; if one ball costs 9 cents for 81 cents there can be bought as many balls as 9 is contained into 81 which are 9 balls.

Written solution.

II. 1. 36 cents=cost of 4 balls.
2. 9 cents=36 cents÷4=cost of 1 ball.
3. 81 cents=cost of 81÷9, or 9 balls.

III. ∴ If 4 balls cost 36 cents, for 81 cents there can be bought 9 balls.

I. What number divided by $\frac{3}{5}$ will give 10 for a quotient?

II. 1. $\frac{5}{5}$=the number.
2. $\frac{5}{5}\div\frac{3}{5}=\frac{5}{5}\times\frac{5}{3}=\frac{5}{3}$=quotient
3. 10=quotient.
4. ∴ $\frac{5}{3}=10$,
5. $\frac{1}{3}=\frac{1}{5}$ of 10=2, and
6. $\frac{3}{3}$=3 times 2=6=the number.

III. ∴ 6=the number required.

I. \$25 is $\frac{3}{5}$ of the cost of a barrel of wine; what did it cost?

II. 1. $\frac{5}{5}$=cost of the wine per barrel.
2. $\frac{3}{5}$ of cost=\$24,
3. $\frac{1}{5}$ of cost=$\frac{1}{3}$ of \$24=\$8,
4. $\frac{5}{5}$ of cost=5 times \$8=\$40,

III. ∴ \$40=cost of wine.

I. What number is that from which, if you take $\frac{3}{7}$ of itself, the remainder will be 16?

II. 1. $\frac{7}{7}$=the number.
2. $\frac{7}{7}-\frac{3}{7}=\frac{4}{7}$=remainder after taking away $\frac{3}{7}$.
3. 16=remainder.
4. ∴ $\frac{4}{7}=16$,
5. $\frac{1}{7}=\frac{1}{4}$ of 16=4, and
6. $\frac{7}{7}$=7 times 4=28=the number.

III. ∴ 28=the required number.

I. A boat is worth \$900; a merchant owns $\frac{5}{8}$ of it, and sells $\frac{1}{3}$ of his share; what part has he left, and what is it worth?

II.
- A.
  1. $\frac{5}{8}$=part the merchant owned.
  2. $\frac{1}{3}$ of $\frac{5}{8}=\frac{5}{24}$=part he sold.
  3. $\therefore \frac{5}{8}-\frac{5}{24}=\frac{15}{24}-\frac{5}{24}=\frac{10}{24}=\frac{5}{12}$=part he had left.
- B.
  1. $\$900$=value of $\frac{12}{12}$, or the whole ship.
  2. $\$75=\frac{1}{12}$ of $\$900$=value of $\frac{1}{12}$ of the ship.
  3. $\$375=5$ times $\$75$=value of $\frac{5}{12}$ of the ship, or part he had left.

III. $\therefore$ $\frac{5}{12}$=part he had left, and $\$375$=value of it.

I. A and B were playing cards. B lost \$14, which was $\frac{7}{10}$ times $\frac{2}{3}$ as much as A then had; and when they commenced, $\frac{5}{8}$ of A's money equaled $\frac{2}{7}$ of B's. How much had each when they began to play?

II.
(1.) $\frac{5}{8}$ of A's money=$\frac{2}{7}$ of B's.
(2.) $\frac{1}{8}$ of A's money=$\frac{1}{5}$ of $\frac{2}{7}=\frac{2}{35}$ of B's.
(3.) $\frac{8}{8}$ of A's money=8 times $\frac{2}{35}=\frac{16}{35}$ of B's.
(4.) $\frac{35}{35}$=B's money when they began to play. Then
(5.) $\frac{16}{35}$=A's money when they began.
(6.)
  1. $\frac{15}{15}$=A's money after winning $\$14$ from B.
  2. $\$14$=what B lost.
  3. $\frac{7}{10}$ times $\frac{2}{3}=\frac{7}{15}$=part A's money is of $\$14$.
  4. $\therefore \frac{7}{15}=\$14$,
  5. $\frac{1}{15}=\frac{1}{7}$ of $\$14=\$2$, and
  6. $\frac{15}{15}=15$ times $\$2=\$30$=A's money after winning $\$14$ from B.

(7.) $\therefore \$30-\$14=\$16$=A's money at first.
(8.) $\therefore \frac{16}{35}=\$16$, from (5),
(9.) $\frac{1}{35}=\frac{1}{16}$ of $\$16=\$1$, and
(10.) $\frac{35}{35}=35$ times $\$1=\$35$=B's money at first.

III. $\therefore$ $\$16$=A's money at first, and $\$35$=B's money at first.

(*Stod. Int. A., p. 111, prob. 30.*)

I. A drover being asked how many sheep he had, said, if to $\frac{1}{3}$ of my flock you add the number $9\frac{1}{2}$, the sum will be $99\frac{1}{2}$; how many sheep had he?

II.
1. $\frac{3}{3}$=the number of sheep.
2. $\frac{1}{3}+9\frac{1}{2}=\frac{1}{3}$ of the number$+9\frac{1}{2}$.
3. $99\frac{1}{2}=\frac{1}{3}$ of the number$+9\frac{1}{2}$.
4. $\therefore \frac{1}{3}+9\frac{1}{2}=99\frac{1}{2}$ or
5. $\frac{1}{3}=99\frac{1}{2}-9\frac{1}{2}=90$, and
6. $\frac{3}{3}=3$ times $90=270$=number of sheep.

III. $\therefore$ He had 270 sheep.

I. Heman has 6 books more than Handford, and both have 26; how many have each?

II.
1. $\frac{2}{2}$=number Handford has. Then
2. $\frac{2}{2}+6$=Heman's number.
3. $\frac{2}{2}+\frac{2}{2}+6=\frac{4}{2}+6$=number both have.
4. 26=number both have.
6. $\therefore$ $\frac{4}{2}+6=26$ or
5. $\frac{4}{2}=26-6=20$.
7. $\frac{1}{2}=\frac{1}{4}$ of $20=5$, and
8. $\frac{2}{2}=2$ times $5=10$=Handford's number.
9. $\frac{2}{2}+6=16$=Heman's number.

III. $\therefore$ Handford had 10 books, and Heman had 16 books. (*Stod. Int. A., p. 116, prob. 2.*)

I. A man and his wife can drink a keg of wine in 6 days, and the man alone in 10 days; how many days will it last the woman?

II.
1. 6 days=time it takes both to drink it.
2. $\frac{1}{6}$=part they drink in one day.
3. 10 days=time it takes the man to drink it.
4. $\frac{1}{10}$=part he drinks in one day.
5. $\therefore$ $\frac{1}{6}-\frac{1}{10}=\frac{5}{30}-\frac{3}{30}=\frac{1}{15}$=part the woman drinks in one day.
6. $\frac{15}{15}$=what the woman drinks in $\frac{15}{15}\div\frac{1}{15}=15$ days.

III. $\therefore$ It will take the woman 15 days.

(*R. Alg. I., p. 112, prob. 59.*)

I. A man was hired for 80 days, on this condition: that for every day he worked he should receive 60 cents, and for every day he was idle he should forfeit 40 cents. At the expiration of the time, he received \$40. How many days did he work?

II.
1. \$.60=what he receives a day.
2. \$48=80×\$.60=what he would have received had he worked the whole time.
3. \$40=what he received.
4. $\therefore$ \$48—\$40=\$8=what he lost by his idleness.
5. \$1=\$.60, his wages,+\$.40, what he had to forfeit,= what he lost a day.
6. $\therefore$ \$8=what he lost in 8÷1, or 8 days.
7. 80 days—8 days=72 days, the time he worked.

III. $\therefore$ He worked 72 days.

I. A ship-mast 51 feet high, was broken off in a storm, and $\frac{2}{3}$ of the length broken off, equaled $\frac{3}{4}$ of the length remaining; how much was broken off, and how much remained?

II.
1. $\frac{2}{3}$ of length broken off$=\frac{3}{4}$ of length remaining,
2. $\frac{1}{3}$ of length broken off$=\frac{1}{2}$ of $\frac{3}{4}=\frac{3}{8}$ of length remaining,
3. $\frac{3}{3}$ of length broken off$=3$ times $\frac{3}{8}=\frac{9}{8}$ of length remaining.
4. $\frac{8}{8}$=length remaining.
5. $\frac{9}{8}$=length broken off.
6. $\frac{9}{8}+\frac{8}{8}=\frac{17}{8}$=whole length.
7. 51 feet=whole length.
8. ∴ $\frac{17}{8}=51$ feet,
9. $\frac{1}{8}=\frac{1}{17}$ of 51 feet=3 feet, and
10. $\frac{8}{8}=8$ times 3 feet=24 feet, length remaining.
11. $\frac{9}{8}=9$ times 3 feet=27 feet, length broken off.

III. ∴ 24 feet=length remaining, and
27 feet=length broken off.

I. A boy being asked his age, said, "4 times my age is 24 years more than 2 times my age;" how old was he?

II.
1. $\frac{2}{2}$=his age.
2. $4\times\frac{2}{2}=\frac{8}{2}=4$ times his age.
3. $2\times\frac{2}{2}=\frac{4}{2}=2$ times his age.
4. ∴ $\frac{8}{2}=\frac{4}{2}+24$ years or
5. $\frac{8}{2}-\frac{4}{2}=\frac{4}{2}=24$ years.
6. $\frac{1}{2}=\frac{1}{4}$ of 24 years=6 years, and
7. $\frac{2}{2}=2$ times 6 years=12 years, his age.

III. ∴ He is 12 years old. (*Stod. Int. A., p. 116, prob. 16.*)

I. If 10 men or 18 boys can dig 1 acre in 11 days, find the number of boys whose assistance will enable 5 men to dig 6 acres in 6 days.

II.
1. 1 A.=what 10 men dig in 11 days.
2. $\frac{1}{10}$ A.=what 1 man digs in 11 days.
3. $\frac{1}{110}$ A.$=\frac{1}{11}$ of $\frac{1}{10}$ A.=what 1 man digs in 1 day.
4. $\frac{1}{22}$ A.$=\frac{5}{110}$ A.=5 times $\frac{1}{110}$ A.=what 5 men dig in 1 day.
5. $\frac{3}{11}$ A.$=\frac{6}{22}$ A.=6 times $\frac{1}{22}$ A.=what 5 men dig in 6 days.
6. ∴ 6 A.$-\frac{3}{11}$ A.$=5\frac{8}{11}$ A.=what is to be dug by the boys in 6 days.
7. 1 A.=what 18 boys dig in 11 days.
8. $\frac{1}{18}$ A.=what 1 boy digs in 11 days.
9. $\frac{1}{198}$ A.$=\frac{1}{11}$ of $\frac{1}{18}$ A.=what 1 boy digs in 1 day.
10. $\frac{1}{33}$ A.$=\frac{6}{198}$ A.=6 times $\frac{1}{198}$ A.=what 1 boy digs in 6 days.
11. $5\frac{8}{11}$ A.=what $5\frac{8}{11}\div\frac{1}{33}$, or 189, boys dig in 6 days.

III. ∴ It will take 198 boys.

(*R. 3d p., O. E., p. 318, prob. 66.*)

I. A man after doing $\frac{3}{5}$ of a piece of work in 30 days, calls an assistant; both together complete it in 6 days. In what time could the assistant complete it alone?

II.
1. $\frac{3}{5}$=part the man does in 30 days.
2. $\frac{1}{50}=\frac{1}{30}$ of $\frac{3}{5}$=part he does in 1 day.
3. $\frac{2}{5}=\frac{5}{5}-\frac{3}{5}$=part he and the assistant do in 6 days.
4. $\frac{1}{15}=\frac{1}{6}$ of $\frac{2}{5}$=part he and the assistant do in 1 day.
5. $\therefore\ \frac{1}{15}-\frac{1}{50}=\frac{10}{150}-\frac{3}{150}=\frac{7}{150}$=part the assistant does in 1 day.
6. $\frac{150}{150}$=part the assistant does in $\frac{150}{150}\div\frac{7}{150}=21\frac{3}{7}$ days.

III. $\therefore$ It will take the assistant $21\frac{3}{7}$ days.

(*R. 3d p., O. E., p. 318, prob. 71.*)

Explanation.—Since the man does $\frac{3}{5}$ of the work before he called on the assistant, there remains $\frac{5}{5}-\frac{3}{5}=\frac{2}{5}$, which he and the assistant do in 6 days. Hence they do $\frac{1}{6}$ of $\frac{2}{5}$, or $\frac{1}{15}$ of the work in one day. If the man and his assistant do $\frac{1}{15}$ of the work in 1 day and the man does $\frac{1}{50}$ of the work in 1 day, the assistant does the difference between $\frac{1}{15}$ and $\frac{1}{50}$ which is $\frac{7}{150}$ of the work in 1 day. Hence it will take $\frac{150}{150}\div\frac{7}{150}$, or $21\frac{3}{7}$ days, to do the work.

I. A person being asked the time of day, replied that it was past noon, and that $\frac{3}{4}$ of the time past noon was equal to $\frac{3}{5}$ of the time to midnight. What was the time of day?

II.
1. $\frac{3}{4}$ of the time past noon=$\frac{3}{5}$ of the time to midnight.
2. $\frac{1}{4}$ of the time past noon=$\frac{1}{3}$ of $\frac{3}{5}=\frac{1}{5}$ of the time to midnight.
3. $\frac{4}{4}$, or the time past noon,=4 times $\frac{1}{5}=\frac{4}{5}$ of the time to midnight.
4. $\frac{5}{5}$=time to midnight. Then
5. $\frac{4}{5}$=time past noon.
6. $\frac{4}{5}+\frac{5}{5}=\frac{9}{5}$=time from noon to midnight.
7. 12 hours=time from noon to midnight.
8. $\therefore\ \frac{9}{5}$=12 hours,
9. $\frac{1}{5}=\frac{1}{9}$ of 12 hours=$1\frac{1}{3}$ hours, and
10. $\frac{4}{5}$=4 times $1\frac{1}{3}$ hours=$5\frac{1}{3}$ hours=5 hr. 20 min., time past noon.

III. $\therefore$ It is 20 min. past 5 o'clock, P. M.

(*Milne's Prac. A., p. 360, prob. 47.*)

Note.—From 3, we have the statement that the time past noon is $\frac{4}{5}$ of the time to midnight. Hence, if $\frac{5}{5}$ is the time to midnight, $\frac{4}{5}$ is the time past noon or if $\frac{10}{10}$ is the time to midnight, $\frac{8}{10}$ is the time past noon.

I. A person being asked the time of day, said that $\frac{5}{9}$ of the time past noon equals the time to midnight. What is the time of day?

II.
1. $\frac{7}{7}$=time past noon. Then
2. $\frac{5}{7}$=time to midnight.
3. $\frac{5}{7}+\frac{7}{7}=\frac{12}{7}$=time from noon to midnight.
4. 12 hours=time from noon to midnight.
5. ∴ $\frac{12}{7}$=12 hours.
6. $\frac{1}{7}=\frac{1}{12}$ of 12 hours=1 hour, and
7. $\frac{7}{7}$=7 times 1 hour=7 hours=time past noon.

III. ∴ It is 7 o'clock P. M.

I. A man being asked the hour of day, replied that $\frac{1}{4}$ of the time past 3 o'clock equaled $\frac{1}{2}$ of the time to midnight; what was the hour?

II.
1. $\frac{1}{4}$ of the time past 3 o'clock=$\frac{1}{2}$ of the time to midnight.
2. $\frac{4}{4}$, or the time past 3 o'clock,=4 times $\frac{1}{2}=\frac{4}{2}$ of the time to midnight.
3. $\frac{2}{2}$=time to midnight.
4. $\frac{4}{2}$=time past 3 o'clock.
5. $\frac{4}{2}+\frac{2}{2}=\frac{6}{2}$=time from 3 o'clock to midnight.
6. 9 hours=time from 3 o'clock to midnight.
7. ∴ $\frac{6}{2}$=9 hours.
8. $\frac{1}{2}=\frac{1}{6}$ of 9 hours=$1\frac{1}{2}$ hours, and
9. $\frac{4}{2}$=4 times $1\frac{1}{2}$ hours=6 hours=time past 3 o'clock.
10. $\frac{4}{2}$+3 hours=9 hours, time past noon.

III. ∴ It is 9 o'clock, P. M.

(*Brooks' Int. A., p. 156, prob. 17.*)

I. A person being asked the hour of day, replied, $\frac{2}{3}$ of the time past noon equals $\frac{2}{9}$ of time from now to midnight $+2\frac{2}{3}$ hours; what was the time?

II.
1. $\frac{2}{3}$ of time past noon=$\frac{2}{9}$ of time to midnight$+2\frac{2}{3}$ hours.
2. $\frac{1}{3}$ of time past noon=$\frac{1}{2}$ of $(\frac{2}{9}+2\frac{2}{3}$ hours$)=\frac{1}{9}$ of time to midnight$+1\frac{1}{3}$ hours. [to midnight+4 hours.
3. $\frac{3}{3}$, or time past noon,=3 times $(\frac{1}{9}+1\frac{1}{3}$ hours$)=\frac{1}{3}$ of time
4. $\frac{3}{3}$=time to midnight.
5. $\frac{1}{3}$+4 hours=time past noon. [night.
6. $\frac{3}{3}+\frac{1}{3}$+4 hours=$\frac{4}{3}$+4 hours=time from noon to mid-
7. 12 hours=time from noon to midnight.
8. ∵ $\frac{4}{3}$+4 hours=12 hours.
9. $\frac{4}{3}$=12 hours—4 hours=8 hours,
10. $\frac{1}{3}=\frac{1}{4}$ of 8 hours=2 hours, and
11. $\frac{1}{3}$+4 hours=6 hours=time past noon.

III. ∴ It is 6 o'clock, P. M.

(*Stod. Int. A., p. 128, Prob. 29.*)

I. A father gave to each of his sons $5 and had $30 remaining; had he given them $8 each, it would have taken all his money; required the number of sons.

II.
1. \$8=amount each received by the second condition.
2. \$5=amount each received by the first condition.
3. \$3=\$8—\$5= excess of second condition over first, on each son.
4. ∴ \$30=excess of second condition over first, on 30÷3, or 10 sons.

III. ∴ There were 10 sons.

I. If 50 lb. of sea water contain 2 lb. of salt, how much fresh water must be added to the 50 lb. so that 10 lb. of the new mixture may contain $\frac{1}{3}$ lb. of salt.

II.
1. $\frac{1}{3}$ lb. of salt=what 10 lb. of the new mixture contains.
2. $\frac{3}{3}$, or 1, lb. of salt=what 3 times 10 lb., or 30 lb., of the new mixture contain.
3. 2 lb. of salt=what 2 times 30 lb., or 60 lb., of the new mixture contain.
4. ∴ 60 lb.—50 lb.=10 lb.=quantity of fresh water that must be added.

III. ∴ 10 lb. of fresh water must be added that 10 lb. of the new mixture may contain $\frac{1}{3}$ lb. of salt.

I. A farmer had his sheep in three fields. $\frac{2}{3}$ of the number in the first field equals $\frac{3}{4}$ of the number in the second field, and $\frac{2}{3}$ of the number in the second field equals $\frac{3}{4}$ of the number in the third field. If the entire number was 434, how many were in each field?

II.
(1.)
1. $\frac{2}{3}$ of number in first field=$\frac{3}{4}$ of number in second field.
2. $\frac{1}{3}$ of number in first field=$\frac{1}{2}$ of $\frac{3}{4}$=$\frac{3}{8}$ of number in second field.
3. $\frac{3}{3}$, or number in first field,=3 times $\frac{3}{8}$=$\frac{9}{8}$ of number in second field.

(2.)
1. $\frac{2}{3}$ of number in second field=$\frac{3}{4}$ of number in third field.
2. $\frac{1}{3}$ of number in second field=$\frac{1}{2}$ of $\frac{3}{4}$=$\frac{3}{8}$ of number in third field.
3. $\frac{3}{3}$, or number in second field,=3 times $\frac{3}{8}$=$\frac{9}{8}$ of number in third field.

(3.) $\frac{8}{8}$=number in third field. Then

(4.) $\frac{9}{8}$=number in second field, and

(5.) $\frac{81}{64}$=$\frac{9}{8}$ of number in second field=number in first field in terms of number in third field.

(6.) ∴ $\frac{8}{8}+\frac{9}{8}+\frac{81}{64}=\frac{64}{64}+\frac{72}{64}+\frac{81}{64}=\frac{217}{64}$=number in the three fields.

(7.) 434=number in the three fields.

(8.) ∴ $\frac{217}{64}$=434,

(9.) $\frac{1}{64}=\frac{1}{217}$ of 434=2, and

(10.) $\frac{64}{64}$=64 times 2=128=number of sheep in third field.

(11.) $\frac{72}{64}$=72 times 2=144=number of sheep in second field.

(12.) $\frac{81}{64}$=81 times 2=162=number of sheep in first field.

III. ∴ $\left\{\begin{array}{l}162=\text{number of sheep in first field,}\\144=\text{number of sheep in second field, and}\\128=\text{number of sheep in third field.}\end{array}\right.$

(*Milne's Prac. A., p. 362, prob. 68.*)

I. In a certain school of 80 pupils there are 32 girls; how many boys must leave that there may be 5 boys to 4 girls?

II.
1. 80=whole number of pupils.
2. 32=number of girls.
3. 80—32=48=number of boys.
4. $\frac{4}{4}$=number of girls. Then, since the number of boys are to be to the number of girls as 5 : 4,
5. $\frac{5}{4}$=number of boys. But
6. $\frac{4}{4}$=32.
7. $\frac{1}{4}=\frac{1}{4}$ of 32=8, and
8. $\frac{5}{4}$=5 times 8=40=number of boys.
9. ∴ 48—40=8=number that must leave that there may be 5 boys to 4 girls.

III. ∴ 8 boys must leave that there may be 5 boys to 4 girls.

I. How far may a person ride in a coach, going at the rate of 9 miles per hour, provided he is gone only 10 hours, and walks back at the rate of 6 miles per hour?

II.
1. 9 mi.=distance he can *ride* in 1 hour.
2. 1 mi.=distance he can *ride* in $\frac{1}{9}$ hour.
3. 6 mi.=distance he can *walk* in 1 hour.
4. 1 mi.=distance he can *walk* in $\frac{1}{6}$ hour.
5. ∴ $\frac{1}{9}$ hr.+$\frac{1}{6}$ hr.=$\frac{5}{18}$ hr.=time it takes him to ride 1 mi. and walk back.
6. ∴ 10 hours=time it takes him to ride $10\div\frac{5}{18}$, or 36, mi. [and walk back.

III. ∴ He can ride 36 miles.

I. A hound ran 60 rods before he caught the fox, and $\frac{2}{3}$ of the distance the fox ran before he was caught, equaled the distance he was ahead when they started. How far did the fox run, and how far in advance of the hound was he when the chase commenced?

II.
1. $\frac{3}{3}$=distance the fox ran before he was caught. Then
2. $\frac{2}{3}$=distance he was ahead.
3. $\frac{3}{3}+\frac{2}{3}=\frac{5}{3}$=distance the hound ran to catch the fox.
4. 60 rods=distance the hound ran to catch the fox.
5. ∴ $\frac{5}{3}$=60 rods,
6. $\frac{1}{3}=\frac{1}{5}$ of 60 rods=12 rods, and
7. $\frac{2}{3}$=2 times 12 rods=24 rods=distance the fox was [ahead.
8. $\frac{3}{3}$=3 times 12 rods=36 rods=distance the fox ran before he was caught.

III. $\therefore$ $\begin{cases}\text{24 rods=distance the fox was ahead, and}\\ \text{36 rods=distance he ran before he was caught.}\end{cases}$

I. If $\frac{1}{3}$ of 12 be 3, what will $\frac{1}{4}$ of 40 be?

II. $\begin{cases}\text{1. } \frac{1}{3} \text{ of } 12=4.\\ \text{2. } \frac{1}{4} \text{ of } 40=10. \text{ By supposition}\\ \text{3. } 4=3. \text{ Then}\\ \text{4. } 1=\frac{1}{4} \text{ of } 3=\frac{3}{4}, \text{ and}\\ \text{5. } 10=10 \text{ times } \frac{3}{4}=7\frac{1}{2}.\end{cases}$

III. $\therefore$ $\frac{1}{4}$ of $40=7\frac{1}{2}$, on the supposition that $\frac{1}{3}$ of 12 is 3.

I. Eight men hire a coach; by getting 6 more passengers, the expenses of each were diminished $\$1\frac{3}{4}$; what do they pay for the coach?

II. $\begin{cases}\text{1. } \frac{8}{8}=\text{amount paid for the coach.}\\ \text{2. } \frac{1}{8}=\text{amount 1 man would have had to pay, had there been only 8 men.}\\ \text{3. } \frac{1}{14}=\text{amount 1 man paid since there were 8 men+6 men, or 14 men.}\\ \text{4. } \therefore \frac{1}{8}-\frac{1}{14}=\frac{7}{56}-\frac{4}{56}=\frac{3}{56}=\text{what each saved.}\\ \text{5. } \$1\frac{3}{4}=\text{what each saved.}\\ \text{6. } \therefore \frac{3}{56}=\$1\frac{3}{4},\\ \text{7. } \frac{1}{56}=\frac{1}{3} \text{ of } \$1\frac{3}{4}=\$\frac{7}{12}, \text{ and}\\ \text{8. } \frac{56}{56}=56 \text{ times } \$\frac{7}{12}=\$32\frac{2}{3}=\text{amount paid for the coach.}\end{cases}$

III. $\therefore$ $\$32\frac{2}{3}$=amount paid for the coach.

(*R. H. A., p. 403, prob. 46.*)

Second solution.

II. $\begin{cases}\text{1. } \$1\frac{3}{4}=\text{amount saved by each man.}\\ \text{2. } \$14=8\times\$1\frac{3}{4}=\text{amount saved by the 8 men and paid by the six men.}\\ \text{3. } \therefore \$2\frac{1}{3}=\frac{1}{6} \text{ of } \$14=\text{amount paid by each of the 14 men.}\\ \text{4. } \therefore \$32\frac{2}{3}=14 \text{ times } \$2\frac{1}{3}=\text{amount they paid for the coach.}\end{cases}$

III. $\therefore$ They paid $\$32\frac{2}{3}$ for the coach.

I. For every 10 sheep I keep I plow an acre of land, and allow one acre of pasture for every 4 sheep; how many sheep can I keep on 161 acres?

II. $\begin{cases}\text{1. 1A.=what I plow for every 10 sheep I keep.}\\ \text{2. } \frac{1}{10}\text{A.=what I plow for each sheep I keep.}\\ \text{3. 1A.=what I allow for pasture for every 4 sheep I keep.}\\ \text{4. } \frac{1}{4}\text{A.=what I allow for pasture for each sheep I keep.}\\ \text{5. } \therefore \frac{1}{10}\text{A.}+\frac{1}{4}\text{A.}=\frac{7}{20}\text{A.=land required for every sheep.}\\ \text{6. } \therefore \text{161A.=land required for } 161\div\frac{7}{20}, \text{ or 460 sheep.}\end{cases}$

III. $\therefore$ I can keep 460 sheep on 161 acres.

(*R. Alg. I., p. 112, prob. 64.*)

Complete analysis.

If for every 10 sheep I plow 1 acre, for 1 sheep I plow $\frac{1}{10}$ of an acre; and if for every 4 sheep I pasture 1 acre, for 1 sheep, I

pasture $\frac{1}{4}$ of an acre ; hence 1 sheep requires $\frac{1}{10}$A.$+\frac{1}{4}$A., or $\frac{7}{20}$A., and on 161 A. I could keep as many sheep as $\frac{7}{20}$A. is contained in 161 A., which are 460 sheep.

I. A man was engaged for one year at \$80 and a suit of clothes; he served 7 months, and received for his wages the clothes and \$35; what was the value of the clothes?

II.
1. $\frac{12}{12}$=value of the suit of clothes.
2. $\frac{12}{12}+\$80$=wages for 1 year or 12 months.
3. $\frac{1}{12}+\$6\frac{2}{3}=\frac{1}{12}$ of $(\frac{12}{12}+\$80)$=wages for 1 month.
4. $\frac{7}{12}+\$46\frac{2}{3}=7$ times $(\frac{1}{12}+\$6\frac{2}{3})$=wages for 7 months.
5. $\frac{12}{12}+\$35$=wages for 7 months.
6. $\therefore \frac{12}{12}+\$35=\frac{7}{12}+\$46\frac{2}{3}$.
7. $\frac{5}{12}=\$11\frac{2}{3}$,
8. $\frac{1}{12}=\frac{1}{5}$ of $\$11\frac{2}{3}=\$2\frac{1}{3}$, and
9. $\frac{12}{12}=12$ times $\$2\frac{1}{3}=\$28$=value of suit of clothes.

III. $\therefore$ The suit of clothes is worth \$28.

I. A lady has two silver cups, and only one cover. The first cup weighs 8 ounces. The first cup and cover weighs 3 times as much as the second cup; and the second cup and cover 4 times as much as the first cup. What is the weight of the second cup and the cover?

II.
1. 3 times weight of second cup=weight of cover+weight of first cup, or 8 oz.
2. 1 times weight of second cup=$\frac{1}{3}$ of weight of cover+$2\frac{2}{3}$ oz.
3. $\frac{3}{3}$=weight of cover. Then
4. $\frac{1}{3}+2\frac{2}{3}$ oz.=weight of second cup.
5. $\frac{3}{3}+\frac{1}{3}+2\frac{2}{3}$ oz.$=\frac{4}{3}+2\frac{2}{3}$ oz. = weight of second cup and cover.
6. 32 oz.=4 times 8 oz.=weight of second cup and cover, by the conditions of the problem.
7. $\therefore \frac{4}{3}+2\frac{2}{3}$ oz.=32 oz.
8. $\frac{4}{3}=32$ oz.—$2\frac{2}{3}$ oz.$=29\frac{1}{3}$ oz.
9. $\frac{1}{3}=\frac{1}{4}$ of $29\frac{1}{3}$ oz.$=7\frac{1}{3}$ oz.
10. $\frac{3}{3}=3$ times $7\frac{1}{3}$ oz.=22 oz.=weight of cover.
11. $\frac{1}{3}+2\frac{2}{3}$ oz.$=7\frac{1}{3}$ oz.$+2\frac{2}{3}$ oz.= 10 oz. = weight of second cup.

III. $\therefore$ 22 oz.=weight of cover, and 10 oz.=weight of second cup.

I. A steamboat that can run 15 mi. per hr. with the current and 10 mi. per hr. against it, requires 25 hr. to go from Cincinnati to Louisville and return; what is the distance between the cities?

II.
1. 15 mi.=distance the boat can travel down stream in 1 hour.
2. 1 mi.=distance the boat can travel down stream in $\frac{1}{15}$ hour.
3. 10 mi.=distance the boat can travel up stream in 1 hr.
4. 1 mi.=distance the boat can travel up stream in $\frac{1}{10}$ hr.
5. $\therefore$ $\frac{1}{15}$ hr.+$\frac{1}{10}$ hr.=$\frac{1}{6}$ hr.=time required for the boat to travel 1 mi. down and return.
6. $\therefore$ 25 hr.=time required for the boat to travel 25÷$\frac{1}{6}$, or 150, miles down and return.

III. $\therefore$ The distance between the two places is 150 miles.

I. A, B, and C dine on 8 loaves of bread; A furnishes 5 loaves; B, 3 loaves; C pays the others 8d. for his share; how must A and B divide the money?

II.
1. 8 loaves=what they all eat.
2. $2\frac{2}{3}$ loaves=what each eats.
3. $\therefore$ 5 loaves—$2\frac{2}{3}$ loaves=$2\frac{1}{3}$ loaves=what A furnished towards C's dinner.
4. $\therefore$ 3 loaves—$2\frac{2}{3}$ loaves=$\frac{1}{3}$ loaf=what B furnished towards C's dinner.
5. $\therefore$ $\dfrac{2\frac{1}{3}}{2\frac{2}{3}}=\frac{7}{8}$=A's share, and
6. $\therefore$ $\dfrac{\frac{1}{3}}{2\frac{2}{3}}=\frac{1}{8}$=B's share.
7. $\frac{7}{8}$ of 8d.=7d.=what A should receive, and
8. $\frac{1}{8}$ of 8d.=1d.=what B should receive.

III. $\therefore$ A should receive 7d., and B should receive 1d. (*R. H. A., p. 403, prob. 42.*)

I. A and B dig a ditch 100 rods long for \$100; how many rods does each dig, if they each receive \$50, and A digs at \$.75 per rod, and B at \$1.25?

There has been a vast amount of quibbling about this problem; but a few moments consideration should suffice to settle all dispute, and pronounce upon it the sentence of absurdity.

We have given, the whole amount each received and the amount each received per rod. Hence, if we divide the whole amount each received by the cost per rod, it must give the number of rods he digs. But by doing this we receive 50÷.75, or $66\frac{2}{3}$ rods, what A digs and 50÷1.25, or 40 rods, what B digs, or $106\frac{2}{3}$ rods which is the length of the ditch, and not 100 rods as stated in the problem. The length of the ditch is a function of the cost per rod and the whole cost, and when they are given the length of the ditch is determined. We might propose a problem just as absurd by requiring the circumference of a circle whose area is 1 acre, and diameter 20 rods. Since the area and circumference are functions of the diameter, when either

of these are given, the other is determined and should not be limited to an inaccurate statement.

If, in the original problem, A's price per rod increases at a constant ratio so that when the ditch is completed he is receiving \$1 per rod, and B's price constantly decreases until when the ditch is completed he is receiving \$1 per rod, then the problem is solvable, and the result is 50 rods each.

I. A is 30 years old, and B is 6 years old; in how many years will A be only 4 times as old as B?

II.
1. $\frac{2}{2}$=B's age at the required time. Then
2. $\frac{8}{2}$=A's age at the required time.
3. $\frac{8}{2}-\frac{2}{2}=\frac{6}{2}$=difference of their ages.
4. 30 years—6 years=24 years=difference of their ages.
5. $\therefore$ $\frac{6}{2}$=24 years.
6. $\frac{1}{2}=\frac{1}{6}$ of 24 years=4 years.
7. $\frac{2}{2}$=2 times 4 years=8 years, B's age at the required time.
8. $\therefore$ 8 years—6 years=2 years=the number of years hence when A will be only 4 times as old as B.

III. $\therefore$ In 2 years A will be only 4 times as old as B.

I. Jacob is twice as old as his son who is 20 years of age; how long since Jacob was 5 times as old as his son?

II.
1. 20 years=son's age at present. Then
2. 40 years=Jacob's age at present.
3. $\frac{2}{2}$=son's age at required time. Then
4. $\frac{10}{2}$=Jacob's age at required time.
5. $\therefore$ $\frac{10}{2}-\frac{2}{2}=\frac{8}{2}$=difference of their ages.
6. 40 years—20 years=20 years=difference of their ages.
7. $\therefore$ $\frac{8}{2}$=20 years,
8. $\frac{1}{2}=\frac{1}{8}$ of 20 years=$2\frac{1}{2}$ years, and
9. $\frac{2}{2}$=2 times $2\frac{1}{2}$ years=5 years, son's age at the required time.
10. $\therefore$ 20 years—5 years=15 years=time since Jacob was 5 times as old as his son.

III. $\therefore$ 15 years ago Jacob was 5 times as old as his son.

Remarks.—Observe that the difference between any two persons' ages is constant, that is, if the difference between A's and B's ages is 7 years now, it will be the same in any number of years from now; for, as a year is added to one's age, it is likewise added to the other's age. But the ratio of their ages is constantly changing as time goes on. If A is 3 years old and B 5 years old, A is now $\frac{3}{5}$ as old as B; but in 1 year, A's age will be 4 years and B's 6 years; A is then $\frac{2}{3}$ as old as B. In 7 years, A will be 10 years old and B 12; A will then be $\frac{10}{12}$, or $\frac{5}{6}$, as old as B, and so on. The ratio of any two persons' ages approaches unity as its limit.

I. A fox is 50 leaps ahead of a hound, and takes 4 leaps in the same time that the hound takes 3; but 2 of the hound's leaps equal 3 of the fox's leaps. How many leaps must the hound take to catch the fox?

II.
1. 2 leaps of hound's=3 leaps of fox's.
2. 1 leap of hound's=$\frac{1}{2}$ of 3 leaps=$1\frac{1}{2}$ leaps of the fox's.
3. 3 leaps of hound's=3 times $1\frac{1}{2}$ leaps=$4\frac{1}{2}$ leaps of fox's.
4. ∴ $4\frac{1}{2}$ leaps—4 leaps=$\frac{1}{2}$ leap=what the hound gains in taking 3 leaps.
5. ∴ 1 leap=2 times $\frac{1}{2}$ leap=what the hound gains in taking 6 leaps.
6. ∴ 50 leaps=what the hound gains in taking 50×6 leaps, or 300 leaps.

III. ∴ The hound must take 300 leaps to catch the fox.

Remark —We see that 3 of the hound's leaps equals $4\frac{1}{2}$ leaps of the fox's, But while the hound takes 3 leaps, the fox takes 4 leaps; hence the hound gains $4\frac{1}{2}$—4, or $\frac{1}{2}$, leap of the fox's. But he has 50 leaps of the fox's to gain, and since he gains $\frac{1}{2}$ leap of the fox's in 3 leaps, he must take 300 leaps to gain 50 leaps.

I. If 6 sheep are worth 2 cows, and 10 cows are worth 5 horses; how many sheep can you buy for 3 horses?

II.
1. Value of 2 cows=value of 6 sheep.
2. Value of 1 cow=value of 3 sheep.
3. Value of 10 cows=value of 30 sheep. But 10 cows are worth 5 horses,
4. ∴ Value of 5 horses=value of 30 sheep.
5. Value of 1 horse=value of 6 sheep.
6. Value of 3 horses=value of 18 sheep.

III. ∴ 3 horses are worth 18 sheep.

I. A teacher agreed to teach a certain time upon these conditions: if he had 20 scholars he was to receive $25; but if he had 30 scholars, he was to receive but $30. He had 29 scholars. Required his wages.

II.
1. $25=his rate of wages for 20 pupils.
2. $1.25=$\frac{1}{20}$ of $25=his rate of wages for 1 pupil.
3. $30=his rate of wages for 30 pupils.
4. $1=$\frac{1}{30}$ of $30=his rate of wages for 1 pupil.
5. ∴ $1.25—$1.00=$.25=reduction per pupil by the addition of 10 pupils.
6. $.025=$.25÷10=reduction per pupil by the addition of 1 pupil.
7. $.225=9 times $.025=reduction per pupil by the addition of 9 pupils.
8. ∴ $1.25—$.225=$1.025=his rate of wage per pupil.
9. $29.725=29 times $1.025=his wages for 29 pupils.

III. ∴ His wages were $29.725.

(*Mattoon's Arith., p. 385, prob. 200.*)

Note.—This problem is really indeterminate, because there is no definite rate of increase of wages given for each additional scholar. We might say, since the wages were increased $5 by the addition of 10 scholars, they would be increased $.50 by the addition of one scholar and, consequently, $4.50 by the addition of 9 scholars. Hence, his wages should be $25+$4.50, or $29.50. By assuming different relations between the increase of wages and additional scholars, other results may be obtained. The above solution seems to be the most satisfactory.

I. A gold and silver watch were bought for $\$160$; the silver watch cost only $\frac{1}{7}$ as much as the gold one; how much was the cost of each?

II.
1. $\frac{7}{7}$=cost of the gold watch. Then
2. $\frac{1}{7}$=cost of the silver watch.
3. $\frac{7}{7}+\frac{1}{7}=\frac{8}{7}$=cost of both.
4. $\$160$=cost of both.
5. $\therefore\ \frac{8}{7}=\$160$,
6. $\frac{1}{7}=\frac{1}{8}$ of $\$160=\$20$=cost of the silver watch, and
7. $\frac{7}{7}$=7 times $\$20=\$140$=cost of the gold watch.

III. $\therefore$ $\$20$=cost of the silver watch, and
$\$140$=cost of gold the watch.

I. A man has two watches, and a chain worth $\$20$; if he put the chain on the first watch it will be worth $\frac{2}{3}$ as much as the second watch, but if he put the chain on the second watch it will be worth $2\frac{3}{4}$ times the first watch what is the value of each watch?

II.
1. $\frac{2}{3}$ s.=$\frac{2}{2}$ f.+$\$20$.
2. $\frac{1}{3}$ s.=$\frac{1}{2}$ of ($\frac{2}{2}$ f.+$\$20$)=$\frac{1}{2}$ f.+$\$10$.
3. $\frac{3}{3}$ s.=3 times ($\frac{1}{2}$ f.+$\$10$)=$\frac{3}{2}$ f.+$\$30$. [lem.
4. $\frac{3}{3}$ s.=$\frac{11}{4}$ f.—$\$20$, by the second condition of the prob-
5. $\therefore$ $\frac{11}{4}$ f.—$\$20$=$\frac{3}{2}$ f.+$\$30$, whence
6. $\frac{11}{4}$ f.—$\frac{3}{2}$ f.=$\$30+\$20$, or
7. $\frac{5}{4}$ f.=$\$50$.
8. $\frac{1}{4}$ f.=$\frac{1}{5}$ of $\$50=\$10$, and
9. $\frac{4}{4}$ f.=4 times $\$10=\$40$=value of first watch.
10. $\frac{3}{3}$ s.=$\frac{3}{2}$ f.+$\$30$=$\frac{3}{2}$ of $\$40+\$30=\$90$=value of the second watch.

III. $\therefore$ $\$40$=value of first watch, and
$\$90$=value of second watch.

(*White's Comp. Arith., p. 243, prob. 60*)

I. At the time of marriage a wife's age was $\frac{3}{5}$ of the age of her husband, and 10 years after marriage her age was $\frac{7}{10}$ of the age of her husband; how old was each at the time of marriage?

II.
1. $\frac{5}{5}$=husband's age at the time of marriage. Then
2. $\frac{3}{5}$=wife's age at the time of marriage.
3. $\frac{5}{5}$+10 years=husband's age 10 years after marriage.
4. $\frac{3}{5}$+10 years=wife's age 10 years after marriage. But
5. $\frac{7}{10}$+7 years=$\frac{7}{10}$ of ($\frac{5}{5}$+10 years)=wife's age 10 years after marriage, by second condition of the problem.
6. $\therefore$ $\frac{7}{10}$+7 years=$\frac{3}{5}$+10 years. Whence
7. $\frac{7}{10}-\frac{3}{5}$=10 years—7 years, or
8. $\frac{1}{10}$=3 years. [of marriage.
9. $\frac{10}{10}$=10 times 3 years=30 years=husband's age at time
10. $\frac{3}{5}$, or $\frac{6}{10}$,=6 times 3 years=18 years=wife's age at the time of marriage.

III. ∴ $\begin{cases} 30 \text{ years}=\text{husband's age at time of marriage, and} \\ 18 \text{ years}=\text{wife's age at time of marriage.} \end{cases}$

(*White's Comp. A., p. 241, prob. 35.*)

I. Ten years ago the age of A was $\frac{3}{4}$ of the age of B, and ten years hence the age of A will be $\frac{5}{6}$ of the age of B; find the age of each.

II.
1. $\frac{4}{4}$=B's age 10 years ago. Then
2. $\frac{3}{4}$=A's age 10 years ago.
3. $\frac{4}{4}+10$ years=B's age now, and
4. $\frac{3}{4}+10$ years=A's age now.
5. $\frac{4}{4}+20$ years=B's age 10 years hence, and
6. $\frac{3}{4}+20$ years=A's age 10 years hence.
7. $\frac{5}{6}$ of $(\frac{4}{4}+20$ years$)=\frac{5}{6}+16\frac{2}{3}$ years=A's age 10 years hence.
8. ∴ $\frac{5}{6}+16\frac{2}{3}$ years$=\frac{3}{4}+20$ years; whence
9. $\frac{5}{6}-\frac{3}{4}=20$ years$-16\frac{2}{3}$ years, or
10. $\frac{1}{12}=3\frac{1}{3}$ years, and
11. $\frac{12}{12}=12$ times $3\frac{1}{3}$ years=40 years=B's age 10 years ago.
12. $\frac{3}{4}=\frac{9}{12}=9$ times $3\frac{1}{3}$ years=30 years=A's age 10 years ago.
13. ∴ $\frac{12}{12}+10$ years=50 years=B's age now, and
14. $\frac{9}{12}+10$ years=40 years=A's age now.

III. ∴ $\begin{cases} 50 \text{ years}=\text{B's age, and} \\ 40 \text{ years}=\text{A's age.} \end{cases}$

I. Two men start from two places 495 miles apart, and travel toward each other; one travels 20 miles a day, and the other 25 miles a day; in how many days will they meet?

II.
1. $\frac{2}{2}$=number of days.
2. 20 mi.=distance first travels in 1 day.
3. $\frac{2}{2}\times20$ mi.=distance first travels in $\frac{2}{2}$ days.
4. 25 mi.=distance second travels in 1 day.
5. $\frac{2}{2}\times25$ mi.=distance second travels in $\frac{2}{2}$ days.
6. ∴ $\frac{2}{2}\times20$ mi.$+\frac{2}{2}\times25$ mi.$=\frac{2}{2}\times(20$ mi.$+25$ mi.$)$=distance both travel.
7. 495 mi.=distance both travel.
8. ∴ $(20$ mi.$+25$ mi.$)\times\frac{2}{2}=(45$ mi.$)\times\frac{2}{2}=495$ mi. Whence
9. $\frac{2}{2}=495\div45=11$=number of days.

III. ∴ They will meet in 11 days.

Second solution.

II.
1. 20 miles=distance first travels in a day.
2. 25 miles=distance second travels in a day.
3. ∴ 45 miles=distance both travel in a day.
4. ∴ 495 miles=distance both travel in $495\div45$, or 11, days.

III. ∴ They will meet in 11 days.

Third solution—the one usually given in the schoolroom.

$$\begin{array}{r l} 20+25=45)\!\!\!\! & 495(11 \text{ days.} \\ & \underline{45} \\ & \;\;45 \\ & \;\;\underline{45} \end{array}$$

I. Find a number whose square root is 25 times its cube root.

II.
1. $\frac{2}{2}$=square root of the number. Then
2. $\frac{2}{2}\times\frac{2}{2}$=the number, because the square root×the square root equals the number.
3. $\frac{3}{3}$=the cube root of the number. Then
4. $\frac{3}{3}\times\frac{3}{3}\times\frac{3}{3}$=the number. But
5. $\frac{2}{2}=5\times(\frac{3}{3})$. Hence, squaring both sides,
6. $\frac{2}{2}\times\frac{2}{2}=25\times(\frac{3}{3}\times\frac{3}{3})$. But
7. $\frac{2}{2}\times\frac{2}{2}$=the number, and
8. $\frac{3}{3}\times\frac{3}{3}\times\frac{3}{3}$=the number.
9. ∴ $\frac{3}{3}\times\frac{3}{3}\times\frac{3}{3}=25\times(\frac{3}{3}\times\frac{3}{3})$. Dividing by $(\frac{3}{3}\times\frac{3}{3})$,
10. $\frac{3}{3}=25$.
11. ∴ $(\frac{3}{3})^3=25^3=15625$.

III. ∴ The number is 15625. (*R. H. A., p. 367, prob. 14.*)

I. A man bought a horse, saddle and bridle for \$150; the cost of the saddle was $\frac{1}{6}$ of the cost of the horse, and the cost of the bridle was $\frac{1}{2}$ the cost of the saddle; what was the cost of each?

II.
1. $\frac{12}{12}$=cost of the horse. Then
2. $\frac{2}{12}=\frac{1}{6}$ of $\frac{12}{12}$=cost of the saddle, and
3. $\frac{1}{12}=\frac{1}{2}$ of $\frac{2}{12}$=cost of the bridle.
4. $\frac{15}{12}=\frac{12}{12}+\frac{2}{12}+\frac{1}{12}$=cost of all.
5. \$150=cost of all.
6. ∴ $\frac{15}{12}$=\$150, and
7. $\frac{1}{12}=\frac{1}{15}$ of \$150=\$10=cost of bridle.
8. $\frac{12}{12}$=12 times \$10=\$120=cost of horse.
9. $\frac{2}{12}$=2 times \$10=\$20=cost of saddle.

III. ∴ \$10=cost of the bridle,
\$20=cost of the saddle, and
\$120=cost of the horse.

(*White's Comp. A., p. 241, prob. 39.*)

I. A and B perform $\frac{9}{10}$ of a piece of work in 2 days, when, B leaving, A completes it in $\frac{1}{2}$ day; in what time can each complete it alone?

II.
1. $\frac{9}{10}$=part A and B do in 2 days.
2. $\frac{9}{20}=\frac{1}{2}$ of $\frac{9}{10}$=part A and B do in 1 day.
3. $\frac{10}{10}-\frac{9}{10}=\frac{1}{10}$=part left after B quits, and which A completes in $\frac{1}{2}$ day.
4. $\frac{2}{10}=\frac{1}{5}$=part A can do in 1 day.
5. $\therefore$ $\frac{5}{5}$=part A can do in $\frac{5}{5}\div\frac{1}{5}=5$ days.
6. $\frac{9}{20}-\frac{1}{5}=\frac{5}{20}=\frac{1}{4}$=part B can do in 1 day.
7. $\therefore$ $\frac{4}{4}$=part B can do in $\frac{4}{4}\div\frac{1}{4}$, or 4, days.

III. $\therefore$ A can do the work in 5 days, and B can do the work in 4 days.

(*White's Comp. A., p. 280, prob. 193.*)

I. A and B can do a piece of work in 12 days, B and C in 9 days, and A and C in 6 days; how long will it take each alone to do the work?

II.
1. 12 days=time it takes A and B to do the work.
2. $\therefore$ $\frac{1}{12}$=part they do in 1 day.
3. 9 days=time it takes B and C to do the work.
4. $\therefore$ $\frac{1}{9}$=part they do in 1 day.
5. 6 days=time it takes A and C to do the work.
6. $\therefore$ $\frac{1}{6}$=part they do in 1 day.
7 $\therefore$ $\frac{1}{12}+\frac{1}{9}+\frac{1}{6}=\frac{13}{36}$=part A and B, B and C, and A and C do in 1 day=twice the work A, B, and C do in 1 day.
8. $\therefore$ $\frac{13}{72}=\frac{1}{2}$ of $\frac{13}{36}$=part A, B, and C do in 1 day.
9. $\frac{13}{72}-\frac{1}{12}=\frac{7}{72}$=part A, B, and C do in 1 day—part B and C do in 1 day=part C does in 1 day.
10. $\frac{72}{72}$=part C does in $\frac{72}{72}\div\frac{7}{72}$, or $10\frac{2}{7}$ days.
11. $\frac{13}{72}-\frac{1}{9}=\frac{5}{72}$=part A, B, and C do in 1 day—part B and C do in 1 day=part A does in 1 day.
12. $\frac{72}{72}$=part A does in $\frac{72}{72}\div\frac{5}{72}=14\frac{2}{5}$ days.
13. $\frac{13}{72}-\frac{1}{6}=\frac{1}{72}$=part A, B, and C do in 1 day—part A and C do in 1 day=part B does in 1 day.
14. $\frac{72}{72}$=part B does in $\frac{72}{72}\div\frac{1}{72}=72$ days

III. $\therefore$ $14\frac{2}{5}$ days=time it takes A, 72 days=time it takes B, and $10\frac{2}{7}$ days=time it takes C.

(*White's Comp. A., p. 194, prob. 280.*)

I. The head of a fish is 8 inches long, the tail is as long as the head and $\frac{1}{2}$ of the body+10 inches, and the body is as long as the head and tail; what is the length of the fish?

II.
1. $\frac{2}{2}$=length of body.
2. 8 in.=length of head.
3. $\frac{1}{2}$ l. of b.+10 in.+8 in.=$\frac{1}{2}$ l. of b.+18 in.=length of tail.
4. $\frac{2}{2}$ l. of b.=length of head+length of tail.
5. ∴ $\frac{2}{2}$ l. of b.=($\frac{1}{2}$ l. of b.+18 in.)+8 in.=$\frac{1}{2}$ l. of b.+26 in. Whence
6. $\frac{2}{2}$ l. of b.—$\frac{1}{2}$ l. of b.=$\frac{1}{2}$ l. of b.=26 in.
7. ∴ $\frac{2}{2}$ l. of b., or length of body,=2 times 26 in.=52 in.
8. $\frac{1}{2}$ l. of b.+18 in.=26 in.+18 in.=44 in.=length of tail.
9. ∴ 52 in.+44 in.+8 in.=104 in.=length of the fish.

III. ∴ The length of the fish is 104 inches.

I. Henry Adams bought a number of pigs for $48; and losing 3 of them, he sold $\frac{2}{3}$ of the remainder, minus 2, for cost, receiving $32 less than all cost him; required the number purchased.

II.
1. $\frac{3}{3}$=remainder after losing 3. Then
2. $\frac{3}{3}$+3=number at first.
3. $\frac{2}{3}$ of r.—2=number sold.
4. $48—$32=$16=what was received for $\frac{2}{3}$ of r.—2.
5. $8=$\frac{1}{2}$ of $16=what was received for $\frac{1}{2}$ of ($\frac{2}{3}$ of r.—2), or $\frac{1}{3}$ of r.—1.
6. $24=3 times $8=what was received for 3 times ($\frac{1}{3}$ of r.—1)=$\frac{3}{3}$ of r —3.
7. ∴ $48—$24=$24=what ($\frac{3}{3}$ of r.+3)—($\frac{3}{3}$ of r.—3), or 6 pigs cost.
8. $4=$\frac{1}{6}$ of $24=what 1 pig cost.
9. ∴ $48=what 48÷4, or 12, pigs cost.

III. ∴ He bought 12 pigs.

(*Brooks' Int. A., p. 164, prob. 9.*)

I. A bought some calves for $80; and having lost 10, he sold 4 more than $\frac{2}{3}$ of the remainder for cost and received $32 less than all cost; required the number purchased.

II.
1. $\frac{3}{3}$=remainder after losing 10. Then
2. $\frac{3}{3}$+10=number purchased.
3. $\frac{2}{3}$ of r.+4=number sold. [cost.
4. $80—$32=$48= cost of $\frac{2}{3}$ of r.+4, since they sold at
5. $24=$\frac{1}{2}$ of $48=cost of $\frac{1}{2}$ of ($\frac{2}{3}$ of r.+4)=$\frac{1}{3}$ of r.+2.
6. $72=3 times $24=cost of 3 times ($\frac{1}{3}$ of r.+2)=$\frac{3}{3}$ of r.+6. [cost.
7. ∴ $80—$72=$8=what ($\frac{3}{3}$ of r.+10)—($\frac{3}{3}$ of r.+6), or 4
8. $2=$\frac{1}{4}$ of $8=what 1 cost.
9. $80=what 80÷2, or 40 cost.

III. ∴ He bought 40 calves.

(*Brook's Int. A., p. 164, prob. 10.*)

I. A lost $\frac{3}{5}$ of his sheep; now if he finds 5 and sells $\frac{3}{5}$ of what he then has for cost price, he will receive \$18; but if he loses 5 and sells $\frac{3}{5}$ of the remainder for cost price, he will receive \$6; how many sheep had he at first?

II.
1. $\frac{5}{5}$= the number of sheep he had at first.
2. $\frac{3}{5}$= the number he lost.
3. $\frac{5}{5}-\frac{3}{5}=\frac{2}{5}$, the number he had after losing $\frac{3}{5}$.
4. $\frac{2}{5}+5$= the number he had after finding 5.
5. $\frac{3}{5}$ of $(\frac{2}{5}+5)=\frac{6}{25}+3$, the number he sold.
6. $\frac{3}{5}-5$= the number, had he lost 5.
7. $\frac{3}{5}$ of$(\frac{2}{5}-5)=\frac{6}{25}-3$, the number he would have sold.
8. \$18=what $(\frac{6}{25}+3)$ sheep cost.
9. \$6= what $(\frac{6}{25}-3)$ sheep cost.
10. ∴ \$12=\$18—\$6=what $(\frac{6}{25}+3)$ sheep—$(\frac{6}{25}-3)$ sheep, or 6 sheep cost.
11. \$2=$\frac{1}{6}$ of \$12= what 1 sheep cost.
12. \$18= what 18÷2, or 9 sheep cost. But
13. \$18=what $(\frac{6}{25}+3)$ sheep cost.
14. ∴ $\frac{6}{25}+3$ sheep = 9 sheep, or
15. $\frac{6}{25}$=6 sheep.
16. $\frac{1}{25}=\frac{1}{6}$ of 6 sheep=1 sheep, and
17. $\frac{25}{25}$=25 times 1 sheep =25 sheep.

III. ∴ He had 25 sheep at first.

(*Brook's Int. A., p. 165, prob. 15.*)

I. A man bought a certain number of cows for \$200; had he bought 2 more at \$2 less each, they would have cost him \$216; how many did he buy?

II.
1. \$200=cost of cows.
2. \$216=cost of original number of cows+2 more.
3. \$216—\$200=\$16=cost of 2 cows at \$2 less per head.
4. ∴ \$8=$\frac{1}{2}$ of \$16=cost of 1 cow at \$2 less per head. Then
5. \$8+\$2=\$10=cost of each cow purchased.
6. \$200=cost of 200÷10, or 20 cows.

III. ∴ He bought 20 cows.

(*Brook's Int. A., p. 162, prob. 8.*)

I. A person being asked the hour of day, said, "the time past noon is $\frac{1}{3}$ of the time past midnight;" what was the hour?

II.
1. $\frac{3}{3}$=time past midnight.
2. $\frac{1}{3}$=time past noon.
3. ∴ $\frac{3}{3}-\frac{1}{3}=\frac{2}{3}$=time from midnight to noon.
4. 12 hours=time from midnight to noon.
5. ∴ $\frac{2}{3}$=12 hours.
6. $\frac{1}{3}=\frac{1}{2}$ of 12 hours=6 hours=time past noon.

III. It was 6 o'clock, P. M.

I. Provided the time past 10 o'clock, A. M., equals $\frac{3}{4}$ of the time to midnight; what o'clock is it?

II.
1. $\frac{4}{4}$=time to midnight. Then
2. $\frac{3}{4}$=time past 10 o'clock.
3. $\frac{4}{4}+\frac{3}{4}=\frac{7}{4}$=time from 10 o'clock to midnight.
4. 14 hours=time from 10 o'clock to midnight.
5. $\therefore\ \frac{7}{4}$=14 hours.
6. $\frac{1}{4}=\frac{1}{7}$ of 14 hours=2 hours, and
7. $\frac{3}{4}$=3 times 2 hours=6 hours, time past 10 o'clock=4 o'clock P. M.

III. ∴ It is 4 o'clock, P. M.

I. At what time between 3 and 4 o'clock will the hour and minute hands of a watch be together?

II.
1. $\frac{2}{2}$=distance the h. h. moves past 3. Then
2. $\frac{24}{2}=12\times\frac{2}{2}$=distance the m. h. moves past 12.
3. $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$=distance the m. h. gains on the h. h.
4. 15 min.=distance the m. h. gains on the h. h
5. $\therefore\ \frac{22}{2}$=15 min.
6. $\frac{1}{2}=\frac{1}{22}$ of 15 min.$=\frac{15}{22}$ min.
7. $\frac{24}{2}$=24 times $\frac{15}{22}$ min.$=16\frac{4}{11}$ min.=distance m. h. moves past 12.

III. ∴ It is $16\frac{4}{11}$ min. past 3 o'clock.

Remark.—In problems of this kind, locate the minute hand at 12 and the hour hand at the first of the two numbers between which the conditions of the problem are to be satisfied. Thus in the above problem, at 3 o'clock the minute hand is at 12 and the hour hand at 3.

The minute hand moves over 60 minute spaces while the hour hand moves over 5 minute spaces. Hence the minute hand moves 12 times as fast as the hour hand. Since at 3 o'clock the minute hand is at 12 and the hour hand at 3, and the minute hand moves 12 times as fast as the hour hand, it is evident that the minute hand will overtake the hour hand between 3 and 4. So we let $\frac{2}{2}$=distance the hour hand moves past 3 until it is overtaken by the minute hand. But since the minute hand moves 12 times as fast as the hour hand, while the hour move $\frac{2}{2}$, the minute hand moves 12 times $\frac{2}{2}$, or $\frac{24}{2}$. Now the minute hand has moved from 12 to $3+\frac{2}{2}$, or 15 minutes$+\frac{2}{2}$. Hence the minute hand has gained 15 minutes on the hour hand. It has also gained $\frac{24}{2}-\frac{2}{2}$, or $\frac{22}{2}$. $\therefore\ \frac{22}{2}$=15 minutes.

In solving any problem of this nature, first locate the hands as previously stated, and then ask yourself how far the *minute hand must move* to meet the conditions of the problem, if the *hour hand should remain stationary*.

I. At what time between 6 and 7 o'clock will the minute hand be at right angles with the hour hand?

II.
1. $\frac{2}{2}$=distance h. h. moves past 6.
2. $\frac{24}{2}$=12 times $\frac{2}{2}$=distance m. h. moves past 12.
3. $\therefore\ \frac{24}{2}-\frac{2}{2}=\frac{22}{2}$=distance m. h. gains on h. h.
4. 15 min. or 45 min.=distance m. h. gains on the h. h.
5. $\therefore\ \frac{22}{2}$=15 min. or 45 min.
6. $\frac{1}{2}=\frac{1}{22}$ of 15 min. or $\frac{1}{22}$ of 45 min.$=\frac{15}{22}$ min. or $2\frac{1}{22}$ min.
7. $\frac{24}{2}$=24 times $\frac{15}{22}$ min. or 24 times $2\frac{1}{22}$ min.$=16\frac{4}{11}$ min. or $49\frac{1}{11}$ min.

III. ∴ The minute hand will be at right angles with the hour hand at $16\frac{4}{11}$ min. or $49\frac{1}{11}$ min. past 6 o'clock.

Explanation.—Locate the minute hand at 12 and the hour hand at 6. Now if the hour hand had remained stationary at 6, the minute hand would have to move to 3 or 9, *i. e.*, it would have to gain 15 min. or 45 min. While the minute hand is moving to 3 the hour hand is moving from 6. So the minute hand must move as far past 3 as the hour hand moves past 6. Or while the minute hand is moving to 9 the hour hand is moving past 6. So the minute hand must move as far past 9 as the hour hand is past 6. ∴ The minute hand must gain 15 minutes in the first case and 45 minutes in the second.

I. At what time between 2 and 3 o'clock are the hour and minute hands opposite?

II.
1. $\frac{2}{2}$=distance hour hand moves past 2. Then
2. $\frac{24}{2}$=distance the minute hand moves past 12, in the same time.
3. ∴ $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$=distance minute hand gained on the hour [hand.
4. 40 min.=distance the minute hand gained on the hour hand.
5. ∴ $\frac{22}{2}$=40 min.
6. $\frac{1}{2}=\frac{1}{22}$ of 40 mim.=$1\frac{9}{11}$ min., and
7. $\frac{24}{2}$=24 times $1\frac{9}{11}$ min.=$43\frac{7}{11}$ min.

III. ∴ It is $43\frac{7}{11}$ min. past 2 o'clock when the hands are opposite.

Explanation.—Locate the minute hand at 12 and the hour hand at 2. Now if the hour hand remained stationary at 2, the minute hand would have to move to 8 or over 40 minutes in order to be opposite the hour hand. But while the minute hand is moving to 8, the hour hand is moving from 2. So the minute hand must move as far past 8 as the hour hand is past 2. Since $\frac{2}{2}$ is the distance the hour hand moves past 2, $\frac{2}{2}$ must be the distance the minute hand must move past 8. Hence the distance the minute hand moves is $\frac{2}{2}$+40 min. But $\frac{24}{2}$=distance the minute hand moves. ∴ $\frac{24}{2}=\frac{2}{2}+$ 40 min. or $\frac{22}{2}$=40 min. as shown in step 5.

I. At what time between 3 and 4 o'clock will the minute hand be 5 minutes ahead of the hour hand?

II.
1. $\frac{2}{2}$=distance hour hand moves while the m. h. is moving to be 5 min. ahead.
2. $\frac{24}{2}=12\times\frac{2}{2}$=distance minute hand moves while the h. h. [moves $\frac{2}{2}$.
3. ∴ $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$=distance gained by the minute hand.
4. 15 min.+5 mim.=20 min.=distance gained by the m. h.
5. ∴ $\frac{22}{2}$=20 min.
6. $\frac{1}{2}=\frac{1}{22}$ of 20 min.=$\frac{10}{11}$ min.
7. $\frac{24}{2}$=24 times $\frac{10}{11}$ min.=$21\frac{9}{11}$ min.

III. ∴It is $21\frac{9}{11}$ min. past 3 o'clock.

Explanation.—Locate the minute hand at 12 and the hour hand at 3. Now if the hour hand remained stationary at 3, the minute hand would have to move to 4 in order to be 5 min. ahead. But while the minute hand is moving to 4 the hour hand is moving from 3. Hence the minute hand must move as far past 4 as the hour hand moves past 3. But the hour hand

moves $\frac{2}{2}$ past 3; hence, the minute hand must move $\frac{2}{2}+5$ min. past 4, in all, $\frac{2}{2}+20$ min. Hence, the minute hand gains $(\frac{2}{2}+20$ min.$)-\frac{2}{2}=20$ min. on the hour hand.

Remark.—We always find $\frac{24}{2}$, the distance the minute hand moves, for it indicates the time between any two consecutive hours. The hour hand indicates the hour.

I. At what time between 4 and 5 o'clock do the hands of a clock make with each other an angle of 45°?

II.
1. $\frac{2}{2}$=distance the hour hand moves past 4.
2. $\frac{24}{2}$=distance the minute hand moves past 12.
3. $\therefore \frac{24}{2}-\frac{2}{2}=\frac{22}{2}$= distance the minute hand gains on the hour hand.
4. $12\frac{1}{2}$ min. or $27\frac{1}{2}$ min.=distance gained by minute hand.
5. $\therefore \frac{22}{2}=12\frac{1}{2}$ min. or $27\frac{1}{2}$ min.
6. $\frac{1}{2}=\frac{1}{22}$ of $12\frac{1}{2}$ min. or $\frac{1}{22}$ of $27\frac{1}{2}$ min.$=\frac{25}{44}$ min. or $1\frac{1}{4}$ [min.
7. $\frac{24}{2}$=24 times $\frac{25}{44}$ min. or 24 times $1\frac{1}{4}$ min.$=13\frac{7}{11}$ min. or 30 min.

III. $\therefore$ At $13\frac{7}{11}$ min. past 4 or 30 min. past 4, the hands make an angle of 45° with each other.

Explanation.—Locate the minute hand at 12 and the hour hand at 4. $45°=\frac{1}{8}$ of 360°. $\frac{1}{8}$ of 60 min.$=7\frac{1}{2}$ min. Hence, that the hands make an angle of 45°, the minute hand must be either $7\frac{1}{2}$ minutes behind the hour hand or $7\frac{1}{2}$ min. ahead, Now if the hour hand remained stationary at 4, the minute hand would have to move over $12\frac{1}{2}$ min. or $2\frac{1}{2}$ min. past 2. But while the minute hand is moving this distance, the hour hand is moving past 4. Hence, the minute hand must move as far past $2\frac{1}{2}$ min. past 2 as the hour hand moves past 4, *i. e.*, the minute hand moves $\frac{2}{2}+12\frac{1}{2}$ min. Hence, it gains $(\frac{2}{2}+12\frac{1}{2}$ min.$)-\frac{2}{2}=12\frac{1}{2}$ min. The reasoning for the second result is the same as for the first.

I. At what time between 4 and 5 o'clock is the minute hand as far from 8 as the hour hand is from 3?

II.

A.
1. $\frac{2}{2}$=distance the hour hand moves past 4.
2. $\frac{24}{2}=12$ times $\frac{2}{2}$=distance minute hand moves past 12 in the same time.
3. $\therefore \frac{24}{2}+\frac{2}{2}=\frac{26}{2}$=distance both move.
4. 35 min.=distance both move.
5. $\therefore \frac{26}{2}=35$ min.
6. $\frac{1}{2}=\frac{1}{26}$ of 35 min.$=1\frac{9}{26}$ min.
7. $\frac{24}{2}$=24 times $1\frac{9}{26}$ min.$=32\frac{4}{13}$ min.

B.
1. $\frac{2}{2}$=distance the h. h. moves past 4.
2. $\frac{24}{2}$=distance minute hand moves past 12.
3. $\therefore \frac{24}{2}-\frac{2}{2}=\frac{22}{2}$=distance the minute hand gains.
4. 45 min.=distance the minute hand gains.
5. $\therefore \frac{22}{2}=45$ min.
6. $\frac{1}{2}=\frac{1}{22}$ of 45 min.$=2\frac{1}{22}$ min.
7. $\frac{24}{2}$=24 times $2\frac{1}{22}$ min.$=49\frac{1}{11}$ min.

III. $\therefore$ It is $32\frac{4}{13}$ min. or $49\frac{1}{11}$ min. past 4 o'clock.

(*R. H. A., p. 403, prob. 40.*)

Explanation.—This problem requires two different solutions. Locate the minute hand at 12 and the hour hand at 4. The hour hand is now 5 minutes from 3. If the hour hand remained stationary, the minute hand would have to move to 7 to be 5 minutes from 8. But while the minute hand is moving to 7, the hour hand is moving past 4. Hence the minute hand must stop as far from 7 as the hour hand moves past 4; *i. e.*, if the hour hand moves $\frac{2}{2}$ past 4 the minute hand must stop $\frac{2}{2}$ from 7. Then the hour hand will be 5 minutes$+\frac{2}{2}$ from 3 and the minute hand will be $\frac{2}{2}+5$ minutes from 8. While the hour hand moved $\frac{2}{2}$, the minute hand moved 35 min.$-\frac{2}{2}$ $\therefore \frac{24}{2}=35$ min.$-\frac{2}{2}$, whence $\frac{26}{2}=35$ min. $\therefore$ 35 min.=distance they both move. The second part has been explained in previous problems.

I. At what time between 5 and 6 o'clock is the minute hand midway between 12 and the hour hand? When is the hour hand midway between 4 and the minute hand?

II.

A.

1. $\frac{2}{2}$=distance the hour hand moves past 5.
2. $\frac{24}{2}$=distance the minute hand moves in the same time.
3. $\frac{2}{2}+25$ min.=distance from 12 to the hour hand.
4. $\frac{1}{2}$ of $(\frac{2}{2}+25$ min.$)=\frac{1}{2}+12\frac{1}{2}$ min.=distance minute hand moves.
5. $\therefore \frac{24}{2}=\frac{1}{2}+12\frac{1}{2}$ min.
6. $\frac{24}{2}-\frac{1}{2}=\frac{23}{2}=12\frac{1}{2}$ min.
7. $\frac{1}{2}=\frac{1}{23}$ of $12\frac{1}{2}$ min.$=\frac{25}{46}$ min.
8. $\frac{24}{2}=24$ times $\frac{25}{46}$ min.$=13\frac{1}{23}$ min.

B.

1. $\frac{2}{2}$=distance the hour hand moves past 5.
2. $\frac{24}{2}$=distance the minute hand moves in the same time
3. $\frac{2}{2}+5$ min.=distance the hour hand is from 4.
4. $\frac{4}{2}+10$ min.=2 times $(\frac{2}{2}+5$ min.$)$=distance the minute hand is from 4, since the hour hand is midway between it and 4.
5. 20 min.$+(\frac{4}{2}+10$ min.$)=\frac{4}{2}+30$ min.=distance the the minute hand is from 12.
6. $\therefore \frac{24}{2}=\frac{4}{2}+30$ min., or
7. $\frac{24}{2}-\frac{4}{2}=\frac{20}{2}=30$ min.
8. $\frac{1}{2}=\frac{1}{20}$ of 30 min.$=1\frac{1}{2}$ min.
9. $\frac{24}{2}=24$ times $1\frac{1}{2}$ min.=36 min.

III. $\therefore$ A. It is $13\frac{1}{23}$ min. past 5 o'clock.
B. It is 36 min. past 5 o'clock.

(*R. H. A., p. 403, prob. 41.*)

Explanation.—Locate the minute hand at 12 and the hour hand at 5. If the hour hand remained stationary, the minute hand would have to move over $\frac{1}{2}$ of 25 minutes, or $12\frac{1}{2}$ minutes. But while it is moving over $12\frac{1}{2}$ minutes, the hour hand is moving past 4. Hence, the minute hand will have to move $12\frac{1}{2}$ minutes$+\frac{1}{2}$ of the distance the hour hand moves past 4. Hence $\frac{24}{2}=\frac{1}{2}+12\frac{1}{2}$ minutes, as shown by step 5 of A. In B, if the hour hand remained stationary, the minute hand would have to move over 30 minutes, *i. e.*, to 6, that the hour hand may be midway between it and 4. But while the minute hand is moving to 6 the hour hand is moving past 4. Hence the minute hand must move twice as far past 6 as the hour hand moves past

4. But $\frac{2}{2}$=distance the hour hand moves past 4; hence, $\frac{4}{2}$=distance the minute hand moves past 6. Hence, $\frac{4}{2}$+30 minutes=distance the minute hand moves. ∴ $\frac{24}{2}=\frac{4}{2}+30$ minutes, as shown by step 6 of B.

I. At what time between 3 and 4 o'clock will the minute hand be as far from 12 on the left side of the dial plate as the hour hand is from 12 on the right side?

II.
1. $\frac{2}{2}$=distance the hour hand moves past 3.
2. $\frac{24}{2}$=12 times $\frac{2}{2}$=distance the minute hand moves in the same time.
3. $\frac{24}{2}+\frac{2}{2}=\frac{26}{2}$=distance they both move.
4. 45 min.=distance they both move.
5. ∴ $\frac{26}{2}$=45 min.
6. $\frac{1}{2}=\frac{1}{26}$ of 45 min.=$1\frac{19}{26}$ min.
7. $\frac{24}{2}$=24 times $1\frac{19}{26}$ min.=$41\frac{7}{13}$ min.

III. ∴ It is $41\frac{7}{13}$ min. past 3.

Explanation.—Locate the minute hand at 12 and the hour hand at 3. If the hour hand remained stationary, the minute hand would have to move to 9 to be as far from 12 on the left side of the dial plate as the hour hand is from 12 on the right. But while the minute hand is moving to 9, the hour hand is moving past 3. Hence, the minute hand must stop as far from 9 as the hour hand moves past 3. Hence, it is evident, they both move 45 minutes.

I. A man looked at his watch and found the time to be between 5 and 6 o'clock Within an hour he looked again, and found the hands had changed places. What was the exact time when he first looked?

II.
(1.) $\frac{2}{2}$=distance m. h. was ahead of h. h., or the distance the h. h. moved, since it changed place with the m. h. [the two observations.
(2.) $\frac{24}{2}$=distance the m. h. moved in the time between
(3.) ∴ $\frac{24}{2}+\frac{2}{2}=\frac{26}{2}$=distance they both moved.
(4.) 60 min.=distance they both moved.
(5.) ∴ $\frac{26}{2}$=60 min.
(6.) $\frac{1}{2}=\frac{1}{26}$ of 60 min.=$2\frac{4}{13}$ min. [ahead of h. h.
(7.) $\frac{2}{2}$=2 times $2\frac{4}{13}$ min.=$4\frac{8}{13}$ min.=distance m. h. was
(8.)
1. $\frac{2}{2}$=distance h. h. was past 5, at time of first observation. Then [servation.
2. $\frac{24}{2}$=distance m. h. was past 12 at time of first ob-
3. 25 min.+$\frac{2}{2}$+$4\frac{8}{13}$ min.=$\frac{2}{2}+29\frac{8}{13}$= distance m. h. was past 12 at time of first observation.
4. ∴ $\frac{24}{2}=\frac{2}{2}+29\frac{8}{13}$ min.
5. $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}=29\frac{8}{13}$ min.
6. $\frac{1}{2}=\frac{1}{22}$ of $29\frac{8}{13}$ min.=$1\frac{9}{26}$ min.
7. $\frac{24}{2}$=24 times $1\frac{9}{26}$ min.=$32\frac{4}{13}$ min.

III. ∴ It was $32\frac{4}{13}$ min. past 5 o'clock.

Explanation.—It is clear that the minute hand was ahead of the hour hand at the time of the first observation, or else they could not have exchanged places within an hour. Now, we call the distance from the point where the hour hand was located at first to the point where the minute hand was located first, $\frac{2}{2}$. But in the mean time the hour hand has moved to the position occupied by the minute hand and the minute hand has moved on around the dial to the position occupied by the hour hand, *i. e.*, the hour hand has moved $\frac{2}{2}$ and the minute 12 times $\frac{2}{2}$, or $\frac{24}{2}$. Hence, they both moved $\frac{26}{2}$. They both moved 60 minutes since the hand moved on around the dial to the position occupied by the hour hand and the hour hand moved to the position occupied by the minute hand. ∴ $\frac{26}{2}=60$ min. as shown in step (5.) The remaining part of the solution has been explained in previous problems.

I. At a certain time between 8 and 9 o'clock a boy stepped into the schoolroom, and noticed the minute hand between 9 and 10. He left, and on returning within an hour, he found the hour hand and minute hand had exchanged places. What time was it when he first entered, and how long was he gone?

II. A.
(1.) $\frac{2}{2}$=distance m. h. was ahead of the h. h. or distance it moved.
(2.) $\frac{24}{2}$=distance m. h. moved while the h. h. moved [$\frac{2}{2}$.
(3.) $\frac{24}{2}+\frac{2}{2}=\frac{26}{2}$=distance both moved.
(4.) 60 min =distance both moved.
(5.) ∴ $\frac{26}{2}=60$ min.
(6.) $\frac{1}{2}=\frac{1}{26}$ of 60 min.$=2\frac{4}{13}$ min.
(7.) $\frac{2}{2}=2$ times $2\frac{4}{13}$ min.$=4\frac{8}{13}$ min.=distance m. h. [was ahead
(8.)
1. $\frac{2}{2}$=distance h. h. moved past 8.
2. $\frac{24}{2}$=distance m. h. moved in same time.
3. 40 min $+\frac{2}{2}+4\frac{8}{13}$ min. $=\frac{2}{2}+44\frac{8}{13}$ min. = distance m. h. moved to be $4\frac{8}{13}$ min. ahead.
4. ∴ $\frac{24}{2}=\frac{2}{2}+44\frac{8}{13}$ min.
5. $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}=44\frac{8}{13}$ min.
6. $\frac{1}{2}=\frac{1}{22}$ of $44\frac{8}{13}$ min.$=2\frac{4}{143}$ min.
7. $\frac{24}{2}=24$ times $2\frac{4}{143}$ min. $=48\frac{96}{143}$ min.=time [past 8.

B.
1. $\frac{26}{2}$=distance they both moved.
2. 60 min.=distance they both moved.
3. ∴ $\frac{26}{2}=60$ min.
4. $\frac{1}{2}=\frac{1}{26}$ of 60 min.$=2\frac{4}{13}$ min.
5. $\frac{24}{2}=24$ times $2\frac{4}{13}$ min.$=55\frac{5}{13}$ min.=time he [was gone.

III. ∴
A. It was $48\frac{96}{143}$ min past 8 o'clock when he first entered school room.
B. He was gone $55\frac{5}{13}$ min.

I. Suppose the hour, minute, and second hands of a clock turn upon the same center, and are together at 12 o'clock; how long before the second hand, hour hand, and minute hand respectively, will be midway between the other two hands?

II. A.

1. $\frac{2}{2}$=distance the hour hand moves past 12. Then
2. $\frac{24}{2}$=distance the minute hand moves past 12, and
3. $\frac{1440}{2}$=720 times $\frac{2}{2}$=distance the second hand moves past 12.
4. $\frac{1440}{2}-\frac{24}{2}=\frac{1416}{2}$ = distance from the minute hand to the second hand.
5. $\frac{1440}{2}-\frac{24}{2}=\frac{1416}{2}$ = distance from the second hand to the hour hand.
6. $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$ = distance from the hour hand to the second hand.
7. $\frac{1416}{2}+\frac{1416}{2}+\frac{22}{2}=\frac{2854}{2}$=distance around the dial.
8. 60 seconds=distance around the dial as indicated by one revolution of the s. h.
9. $\therefore \frac{2854}{2}$=60 sec.
10. $\frac{1}{2}=\frac{1}{2854}$ of 60 sec.=$\frac{30}{1427}$ sec.
11. $\frac{1440}{2}$=1440 times $\frac{30}{1427}$ sec.=$30\frac{390}{1427}$ sec. = time when s. h. is midway between the h. h. and m. h.

FIG. 1

II. B.

1. $\frac{2}{2}$=distance the hour hand moves past 12. Then
2. $\frac{24}{2}$=distance the minute hand moves past 12, and
3. $\frac{1440}{2}$=distance the second hand moves past 12.
4. $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$=distance from h. h. to m. h.
5. $\frac{22}{2}$=distance from s. h. to h. h., because the h. h. is midway between them. [12.
6. $\frac{22}{2}-\frac{2}{2}$=distance from s. h. to
7. $\frac{1440}{2}+\frac{20}{2}=\frac{1460}{2}$ = distance around the dial.
8. 60 sec.=distance around the dial.
9. $\therefore \frac{1460}{2}$=60 sec.
10. $\frac{1}{2}=\frac{1}{1460}$ of 60 sec.=$\frac{3}{73}$ sec.
11. $\frac{1440}{2}$=1440 times $\frac{3}{73}$ sec.=$59\frac{13}{73}$ sec.=time when the h. h. is midway between the s. h. and m. h.

FIG. 2.

II. C.

1. $\frac{2}{2}$=distance h. h. moves past 12. Then
2. $\frac{24}{2}$=distance m. h. moves past 12, and
3. $\frac{1440}{2}$=distance s h. moves past 12. [h. to s. h.
4. $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$=distance from h.
5. $\frac{22}{2}$=distance from m h. to s. h. [from 12 to s h.
6. $\frac{2}{2}+\frac{22}{2}+\frac{22}{2}=\frac{46}{2}$ = distance
7. $\frac{1440}{2}-\frac{46}{2}=\frac{1394}{2}$ = distance around the dial. [dial.
8. 60 sec =distance around the

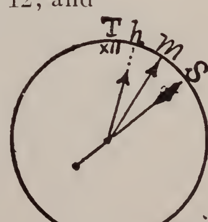


FIG 3.

9. $\therefore \frac{1394}{2}=60$ sec.
10. $\frac{1}{2}=\frac{1}{1394}$ of 60 sec.$=\frac{30}{697}$ sec.
11. $\frac{1440}{2}=1440$ times $\frac{30}{697}$ sec.$=61\frac{683}{697}$ sec.=time past 12 when the m. h. will be midway between the h. h. and s. h.

III. $\therefore$
A. The second hand is midway between h. h. and m. h. at $30\frac{390}{1427}$ sec. past 12.
B. The hour hand is midway between s. h. and m. h. [at $59\frac{13}{73}$ sec. past 12.
C. The minute hand is midway between h. h. and s. h. at $61\frac{683}{697}$ sec. past 12.

Explanation.—A. We represent the distance moved by the hour hand by $\frac{2}{2}$, = the space Th. And since the minute hand moves 12 times as fast as the hour hand, it moves $\frac{24}{2}$. The second hand moves 60 times as fast as the minute hand or 720 times as fast as the hour hand. From T to h is $\frac{2}{2}$ and from I to m is $\frac{24}{2}$. $\therefore$ From h to m is $Tm-Th=\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$. From T to s is $\frac{1440}{2}$. $\therefore$ From m to $s=Ts-Tm=\frac{1440}{2}-\frac{24}{2}=\frac{1416}{2}$. And, by the condition of the problem, the distance from m to s = the distance from m to h. $\therefore$ from m to $h=\frac{1416}{2}+\frac{1416}{2}=\frac{2832}{2}$. We have seen, already, that the distance from h to m is $\frac{22}{2}$. $\therefore$ The whole distance around the dial is $\frac{2832}{2}+\frac{22}{2}=\frac{2854}{2}$.

B. From T to h is $\frac{2}{2}$. From T to m is $\frac{24}{2}$. $\therefore$ From h to $m=Tm-Th=\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$. By the condition of the problem, the distance from h to m=the distance from s to h. $\therefore sT-Th=\frac{22}{2}-\frac{2}{2}=\frac{20}{2}$. From T around the dial to the right of s is $\frac{1440}{2}$. $\therefore$ The whole distance around the dial$=\frac{1440}{2}+\frac{20}{2}=\frac{1460}{2}$.

C. From T to h is $\frac{2}{2}$. From T to m is $\frac{24}{2}$. $\therefore$ From h to $m=\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$. By the condition of the problem, the distance from m to s=the distance from h to $m=\frac{22}{2}$. $\therefore$ From T to s is $\frac{2}{2}+\frac{22}{2}+\frac{22}{2}=\frac{46}{2}$. From T around the dial through T to s is $\frac{1440}{2}$. $\therefore$ The whole distance around the dial is $\frac{1440}{2}-\frac{46}{2}=\frac{1394}{2}$.

I. A sold to B 9 horses and 7 cows for \$300; to C, at the same price, 6 horses and 13 cows, for the same sum: what was the price of each?

II.
1. Cost of 9 horses+cost of 7 cows=\$300. Then the
2. Cost of 36 horses+cost of 28 cows=\$1200, by taking 4 times the number of each.
3. Cost of 6 horses+cost of 13 cows=\$300. Then the
4. Cost of 36 horses+cost of 78 cows=\$1800, by taking 6 times the number of each. But
5. Cost of 36 horses+cost of 28 cows=\$1200.
6. $\therefore$ Cost of 50 cows=\$600, by subtracting; and
7. Cost of 1 cow$=\frac{1}{50}$ of \$600=\$12. The
8. Cost of 7 cows=7 times \$12=\$84.
9. $\therefore$ Cost of 9 horses=\$300—cost of 7 cows=\$300—\$84 =\$216. The
10. Cost of 1 horse$=\frac{1}{9}$ of \$216=\$24.

III. $\therefore$ The cows cost \$12 apiece, and
The horses \$24 apiece.

I. A man at his marriage agreed that if at his death he should leave only a daughter, his wife should have $\frac{3}{4}$ of his estate; and if he should leave only a son she should have $\frac{1}{4}$. He left a son and a daughter. What fractional part of the estate should each receive, and what was each one's portion, if his estate was worth \$6591?

II.
1. $\frac{1}{4}$=daughter's share.
2. $\frac{3}{4}$=wife's share.
3. $\frac{9}{4}$=3 times $\frac{3}{4}$=son's share.
4. $\frac{1}{4}+\frac{3}{4}+\frac{9}{4}=\frac{13}{4}$=whole estate.
5. \$6591=whole estate.
6. ∴ $\frac{13}{4}$=\$6591.
7. $\frac{1}{4}=\frac{1}{13}$ of \$6591=\$507=daughter's share,=$\frac{1}{13}$ of whole estate.
8. $\frac{3}{4}$=3 times \$507=\$1521=wife's share,=$\frac{3}{13}$ of whole estate.
9. $\frac{9}{4}$=9 times \$507=\$4563=son's share,=$\frac{9}{13}$ of whole estate.

III. ∴ \$507=$\frac{1}{13}$ of whole estate=daughter's share.
\$1521=$\frac{3}{13}$ of whole estate=wife's share.
\$4563=$\frac{9}{13}$ of whole estate=son's share.

(*Milne's Prac. A., p. 362, prob. 74.*)

Note.—For a valuable critique, by Marcus Baker, U. S. Coast Survey, on this class of problems, see *School Visitor, Vol. IX., p. 186.*

I. There is coal now on the dock, and coal is running on also from a shoot at a uniform rate. Six men can clear the dock in 1 hour, but 11 men can clear it in 20 minutes; how long would it take 4 men?

II.
1. $\frac{2}{2}$=what one man removes in 1 hour. Then
2. $\frac{12}{2}$=6 times $\frac{2}{2}$=what 6 men remove in 1 hour.
3. $\frac{2}{6}=\frac{1}{3}$ of $\frac{2}{2}$=what 1 man removes in 20 min., or $\frac{1}{3}$ hour.
4. $\frac{22}{6}$=11 times $\frac{2}{6}$=what 11 men remove in $\frac{1}{3}$ hour.
5. ∴ $\frac{12}{2}-\frac{22}{6}=\frac{14}{6}$= what runs on in 1 hr.—$\frac{1}{3}$ hr.=$\frac{2}{3}$ hr. Then
6. $\frac{7}{2}=\frac{14}{6}\div\frac{2}{3}$=what runs on in 1 hour.
7. ∴ $\frac{12}{2}-\frac{7}{2}=\frac{5}{2}$=what was on the dock when the work commenced.
8. $\frac{8}{2}$=what 4 men remove in 1 hour.
9. ∴ $\frac{8}{2}-\frac{7}{2}=\frac{1}{2}$=part of coal removed every hour, that was on the dock at first.
10. $\frac{5}{2}$=coal to be removed in $\frac{5}{2}\div\frac{1}{2}$=5 hours.

(*R. H. A., p. 406, prob. 90.*)

III. ∴ It will take 4 men, 5 hours to clear the dock.

Explanation.—$\frac{12}{2}$=what 6 men remove in 1 hr. and $\frac{22}{6}$=what 11 men removed in $\frac{1}{3}$ hr. In either case the dock was cleared. ∴ $\frac{12}{2}-\frac{22}{6}=\frac{14}{6}$= amount of coal that ran on the dock from the shoot in 1 hr.—$\frac{1}{3}$ hr, or $\frac{2}{3}$ hr. Hence in 1 hr. there will run on, $\frac{14}{6}\div\frac{2}{3}=\frac{7}{2}$. Since $\frac{7}{2}$ run on in 1 hr. and $\frac{12}{2}$=the whole amount of coal removed in 1 hr., $\frac{12}{2}-\frac{7}{2}$, or $\frac{5}{2}$ must be the amount of coal on the dock when the work began. Since $\frac{8}{2}$=the amount 4 men remove in 1 hr. and $\frac{7}{2}$=the amount that runs on the dock in 1 hr., $\frac{8}{2}-\frac{7}{2}$, or $\frac{1}{2}$ is the part of the original quantity removed each hour. Hence, if $\frac{1}{2}$ is removed in 1 hour $\frac{5}{2}$ would be removed in $\frac{5}{2}\div\frac{1}{2}$, or hours.

I. If 12 oxen eat up $3\frac{1}{3}$ acres of pasture in 4 weeks, and 21 oxen eat up 10 acres of like pasture in 9 weeks; to find how many oxen will eat up 24 acres in 18 weeks.

II.
1. 10 parts (say)=what one ox eats in a week. Then
2. 120 parts=12×10 parts=what 12 oxen eat in 1 week,
3. 480 parts=4×120 parts=what 12 oxen eat in 4 weeks.
4. ∴ 480 parts=original grass+growth of grass on $3\frac{1}{3}$ A. in 4 weeks.
5. 144 parts=$\frac{1}{3\frac{1}{3}}$ of 480 parts=original grass+growth of grass on 1 A. in 4 weeks.
6. 210 parts=21×10 parts=what 21 oxen eat in 1 week,
7. 1890 parts=9×210 parts=what 21 oxen eat in 9 weeks.
8. ∴ 1890 parts=original grass+growth of grass on 10 A. in 9 weeks.
9. 189 parts=$\frac{1}{10}$ of 1890 parts=original grass+growth on 1 A in 9 weeks
10. ∴ 189 parts—144 parts=45 parts=growth on 1 A. in 9 weeks—4 weeks, or 5 weeks.
11. 9 parts=$\frac{1}{5}$ of 45 parts=growth on 1 A. in 1 week.
12. 36 parts=4×9 parts=growth on 1 A. in 4 weeks.
13. ∴ 144 parts—36 parts=108 parts=original quantity of grass on 1 A.
14. 2592 parts=24×108 parts=original quantity on 24 A.
15. 216 parts=24×9 parts=growth on 24 A. in 1 week.
16. 3888 parts=18×216 parts=growth on 24 A. in 18 weeks.
17. ∴ 2592 parts+3888 parts=6480 parts=quantity of grass to be eaten by the required oxen.
18. 180 parts=18×10 parts=what 1 ox eats in 18 weeks.
19. ∵ 6480 parts=what $6480\div180$, or 36 oxen eat in 18 weeks.

III. ∴ It will require 36 oxen to eat the grass on 24 A. in 18 weeks.

Note.—This celebrated problem was, very probably, proposed by Sir Isaac Newton and published in his *Arithmetica Universalis* in 1704. Dr. Artemas Martin says, "I have not been able to trace it to any earlier work." For a full treatment of this problem see *Mathematical Magazine*, Vol. 1, No. 2.

I. A man and a boy can mow a certain field in 8 hours, if the boy rests $3\frac{3}{4}$ hours, it takes them $9\frac{1}{2}$ hours. In what time can each do it?

II.
1. $9\frac{1}{2}$ hr.—$3\frac{3}{4}$ hr.=$5\frac{3}{4}$ hr.=time they both work together in the second case.
2. 8 hr.=time it takes them to do the work.
3. ∴ $\frac{1}{8}$=part they do in 1 hour.
4. $\frac{5\frac{3}{4}}{8}=\frac{23}{32}=5\frac{3}{4}$ times $\frac{1}{8}$=part they do in $5\frac{3}{4}$ hours.
5. ∴ $\frac{32}{32}-\frac{23}{32}=\frac{9}{32}$=part the man did in $3\frac{3}{4}$ hours, while the boy rested.
6. ∴ $\frac{3}{40}=\frac{1}{3\frac{3}{4}}$ of $\frac{9}{32}$=part the man did in 1 hour.
7. ∴ $\frac{40}{40}$=part the man can do in $\frac{40}{40}\div\frac{3}{40}$ or $13\frac{1}{3}$ hours.
8. $\frac{1}{8}-\frac{3}{40}=\frac{1}{20}$=part the boy does in one hour.
9. ∴ $\frac{20}{20}$=part the boy can do in $\frac{20}{20}\div\frac{1}{20}$ or 20 hours.

III. ∴ It will take the man $13\frac{1}{3}$ hours, and The boy 20 hours. (*R. H. A., p. 402, prob. 30.*)

I. Six men can do a work in $4\frac{1}{3}$ days; after working 2 days, how many must join them so as to complete it in $3\frac{2}{5}$ days?

II.
1. $4\frac{1}{3}$ days=time it takes 6 men.
2. 26 days=6 times $4\frac{1}{3}$ days=time it takes 1 man.
3. ∴ $\frac{1}{26}$=part 1 man does in 1 day.
4. $\frac{3}{13}$=6 times $\frac{1}{26}$=part 6 men do in 1 day.
5. $\frac{6}{13}$=2 times $\frac{3}{13}$=part 6 men do in 2 days.
6. $\frac{13}{13}-\frac{6}{13}=\frac{7}{13}$=part to be done in $3\frac{2}{5}$ days—2 days, or $1\frac{2}{5}$ days.
7. $\frac{1\frac{2}{5}}{26}=\frac{7}{130}$=part 1 man does in $1\frac{2}{5}$ days.
8. ∴ $\frac{7}{13}$=part $\frac{7}{13}\div\frac{7}{130}$, or 10 men can do in $1\frac{2}{5}$ days.
9. ∴ 10 men—6 men=4 men, the number that must join them.

III. ∴ They must be joined by 4 more men that they may complete the work in $3\frac{2}{5}$ days. *R. H. A., p. 402, prob. 34.*

I. From a ten-gallon keg of wine, one gallon is drawn off and the keg filled with water; if this is repeated 4 times, what will be the quantity of wine in the keg?

II.
1. $\frac{1}{10}$=part drawn out each time.
2. $\frac{9}{10}$=part that was pure wine after the first draught.
3. $\frac{1}{10}$ of $\frac{9}{10}=\frac{9}{100}$=part wine drawn off the second draught.
4. $\frac{9}{10}-\frac{9}{100}=\frac{81}{100}$= part pure wine left after the second draught.
5. $\frac{1}{10}$ of $\frac{81}{100}=\frac{81}{1000}$= part wine drawn off at the third draught.
6. $\frac{81}{100}-\frac{81}{1000}=\frac{729}{1000}$= part pure wine left after the third draught.
7. $\frac{1}{10}$ of $\frac{729}{1000}=\frac{729}{10000}$=part wine drawn off at the fourth draught.
8. $\frac{729}{1000}-\frac{729}{10000}=\frac{6561}{10000}$= part pure wine left after fourth draught.
9. ∴ $\frac{6561}{10000}$ of 10 gal.=6.561 gal.=pure wine left after the fourth draught.

III. ∴ There will be 6.561 gal. of pure wine in the keg after the fourth draught.

I. In the above problem, how many draughts are necessary to draw off half the wine?

II.
1. $\frac{1}{10}$=part wine drawn off at the first draught.
2. $\frac{10}{10}-\frac{1}{10}=\frac{9}{10}$=part wine left after the first draught.
3. $\frac{1}{10}$ of $\frac{9}{10}=\frac{9}{100}=\frac{9}{10^2}$=part wine drawn off at the second draught.
4. $\frac{9}{10}-\frac{9^2}{10^2}=\frac{81}{100}=\left(\frac{9}{10}\right)^2$=part wine left after the second draught.
5. $\frac{1}{10}$ of $\left(\frac{9}{10}\right)^2=\frac{9^2}{10^3}$=part wine drawn off at the third draught.
6. $\left(\frac{9}{10}\right)^2-\frac{9^2}{10^3}=\left(\frac{9}{10}\right)^3$=part wine left after the third draught. By induction,
7. $\left(\frac{9}{10}\right)^n$=part wine left after the nth draught.
8. ∴ $10\left(\frac{9}{10}\right)^n$=number of gal. left after the nth draught.
9. 5=number of gal. left after the nth draught.
10. ∴ $10\left(\frac{9}{10}\right)^n=5$, whence
11. $\left(\frac{9}{10}\right)^n=\frac{1}{2}$. Applying logarithms,
12. $n \log. \frac{9}{10}=\log. \frac{1}{2}$.
13. ∴ $n=\log. \frac{1}{2}\div\log. \frac{9}{10}=.30103\div.\bar{1}.954243=.301030\div.045757=6+$.

III. ∴ In 7 draughts, half and a little more than half of the wine will be drawn off.

PROBLEMS.

1. A man bought a horse and a cow for \$100, and the cow cost $\frac{2}{3}$ as much as the horse; what was the cost of each?
Ans. horse, \$60; cow, \$40.

2. Stephen has 10 cents more than Marthia, and they together have 40 cents; how many have each?
Ans. Stephen, 25¢; Marthia, 15¢.

3. A's fortune added to $\frac{1}{2}$ of B's fortune, equals \$2000; what is the fortune of each, provided A's fortune is to B's as 3 to 4?
Ans. A's, \$1200; B's, \$1600.

4. If 10 oxen eat 4 acres of grass in 6 days, in how many days will 30 oxen eat 8 acres? *Ans.* 4 days.

5. If a 5-cent loaf weighs 7 oz. when flour is worth \$6 a barrel, how much ought it weigh when flour is worth \$7 per barrel? *Ans.*——

6. A lady gave 80 cents to some poor children; to each boy she gave 2 cents, and to each girl 4 cents; how many were there of each, provided there were three times as many boys as girls? *Ans.* 8 girls; 24 boys.

7. Two men or three boys can plow an acre in $\frac{1}{6}$ of a day; how long will it take 3 men and 2 boys to plow it? *Ans.* $\frac{1}{13}$ da.

8. A agreed to labor a certain time for \$60, on the condition that for each day he was idle he should forfeit \$2, at the expiration of the time he received \$30; how many days did he labor, supposing he received \$2 per day for his labor? *Ans.* $22\frac{1}{2}$ days.

9. The head of a fish is 4 inches long, the tail is as long as the head, plus $\frac{1}{2}$ of the body, and the body is as long as the head and tail; what is the length of the fish? *Ans.* 32 inches.

10. In a school of 80 pupils there are 30 girls; how many boys must leave that there may be 3 boys to 5 girls? *Ans.* 38.

11. A steamboat, whose rate of sailing in still water is 12 miles an hour, descends a river whose current is 4 miles an hour and is gone 6 hours; how far did it go? *Ans.* 32 miles.

12. A man keeps 72 cows on his farm, and for every 4 cows he plows 1 acre, and keeps 1 acre of pasture for every 6 cows; how many acres in his farm? *Ans.* 30 acres.

13. A company of 15 persons engaged a dinner at a hotel, but before paying the bill 5 of the company withdrew by which each person's bill was augmented \$$\frac{1}{2}$; what was the bill? *Ans.* \$15.

14. A man sold his horse and sleigh for \$200, and $\frac{4}{5}$ of this is 8 times what his sleigh cost, and the horse cost 10 times as much as the sleigh; required the cost of each. *Ans.* horse, \$200; sleigh, \$20.

15. A went to a store and borrowed as much as he had, and spent 4 cents; he then went to another store and did the same, and then had 4 cents remaining; how much money had he at first? *Ans.* 4 cents.

16. A lady being asked her age, said that if her age were increased by its $\frac{1}{5}$, the sum would equal 3 times her age 12 years ago; what was her age? *Ans.* 20.

17. A lady being asked the hour of day, replied that $\frac{2}{3}$ of the time past noon equaled $\frac{4}{5}$ of the time to midnight, minus $\frac{4}{5}$ of an hour; what was the time? *Ans.* 6 o'clock, P. M.

18. What is the hour of day if $\frac{1}{3}$ of the time to noon equals the time past midnight? *Ans.* 9 o'clock, A. M.

19. A person being asked the time of day, said $\frac{3}{5}$ of the time to midnight equals the time past midnight; what was the time? *Ans.* 9 o'clock, A. M.

20. A traveler on a train notices that $2\frac{1}{4}$ times the number of spaces between the telegraph poles that he passes in a minute is the rate of the train in miles per hour. How far are the poles apart? *Ans.* 198 feet.

21. C's age at A's birth was $5\frac{1}{2}$ times B's age, and now is the sum of A's and B's ages, but if A were now 3 years younger and B 4 years older, A's age would be $\frac{3}{4}$ of B's age. Find their ages. *Ans.* A's, 72 years; B's, 88 years; C's, 160 years.

22. In the above problem change the last *and* to *or*, and what is their ages? *Ans.* A's, 36; B's, 44, and C's, 80.

23. I have four casks, A, B, C, and D respectively. Find the capacity of each, if $\frac{3}{7}$ of A fills B, $\frac{3}{4}$ of B fills C, and C fills $\frac{9}{16}$ of D; but A will fill C and D and 15 quarts remaining. *Ans.* A 35 gal., B 15, C $11\frac{1}{4}$, and D 20.

24. A man and a boy can do a certain work in 20 days: if the boy rests $5\frac{1}{4}$ days it will take them $22\frac{1}{3}$ days; in what time can each do it? *Ans.* The man, 36 da.; the boy, 45 da.

25. A can do a job of work in 40 days, B in 60 days; after both work 3 days, A leaves; when must he return that the work may occupy but 30 days? *Ans.* $22\frac{1}{2}$ days.

26. If 8 men or 15 boys plow a field in 15 days of $9\frac{1}{3}$ hr., how many boys must assist 16 men to do the work in 5 days of 10 hr. each? *Ans.* 12 boys.

27. Bought 10 bu. of potatoes and 20 bu. of apples for \$11; at another time 20 bu. of potatoes and 10 bu. of apples for \$13; what did I pay for each per bu.? *Ans.* Apples 30¢, potatoes 50¢.

28. A farmer sold 17 bu. of barley and 13 bu. of wheat for \$31.55, getting 35¢ a bu. more for wheat than for the barley. Find the price of each per bu. *Ans.* Barley 90¢, wheat \$1.25.

29. After losing $\frac{3}{4}$ of my money I earned \$12; I then spent $\frac{2}{3}$ of what I had and found I had \$36 less than I lost; how much money had I at first? *Ans.* \$60.

30. In a company of 87, the children are $\frac{3}{8}$ of the women, and the women $\frac{4}{9}$ of the men; how many are there of each? *Ans.* 54 men, 24 women, and 9 children.

31. If 4 horses or 6 cows can be kept 10 days on a ton of hay, how long will it last 2 horses and 12 cows? *Ans.* 4 days.

32. A, B, and C buy 4 loaves of bread, A paying 5 cents, B 8 cents, and C 11 cents. They eat 3 loaves and sell the fourth to D for 24 cents. Divide the 24 cents equitably.
Ans. A 5 cents, B 8 cents, and C 11 cents.

33. A and B are at opposite points of a field 135 rods in compass, and start to go around in the same direction, A at the rate of 11 rods in 2 minutes and B 17 rods in 3 minutes. In how many rounds will one overtake the other? *Ans.* B 17 rounds.

34. If a piece of work can be finished in 45 days by 35 men and the men drop off 7 at a time every 15 days, how long will it be before the work is completed? *Ans.* 75 days.

35. A watch which loses 5 min a day was set right at 12 M., July 24th. What will be the true time on the 30th, when the hands of that watch point to 12? *Ans.* 12:30$\frac{30}{287}$ P. M.

36. A seed is planted. Suppose at the end of 3 years it produces a seed, and on each year thereafter each of which when 3 years old produce a seed yearly. All the seeds produced, do likewise; how many seeds will be produced in 21 years?
Ans. 1872.

37. The circumference of a circle is 390 rods. A, B, and C start to go around at the same time. A walks 7 rods per minute, B 13 rods per minute in the same direction; C walks 19 rods per minute in the opposite direction. In how many minutes will they meet? *Ans.* 195 min.

38. If 12 men can empty a cistern into which water is running at a uniform rate, in 40 min., and 15 men can empty it in 30 min., how long will it require 18 men to empty it?
Ans. 24 men.

39. Four men A, B, C, and D, agree to do a piece of work in 130 days. A gets 42d., B 45d., C 48d., and D 51d., for every day they worked, and when they were paid each man has the same amount. How many days did each work?
Ans. A 35$\frac{5885}{7409}$ da., B 33$\frac{3023}{7409}$ da., C 31$\frac{2371}{7409}$ da., and D 29$\frac{3539}{7409}$ da.

40. A fountain has four receiving pipes, A, B, C, and D; A, B, and C will fill it in 6 hours; B, C, and D in 8 hours; C, D, and A in 10 hours; and D, A, and B in 12 hr.: it also has four discharging pipes, W, X, Y, and Z; W, X, and Y will empty it in 6 hours; X, Y, Z in 5 hours; Y, Z, and W in 4 hours; and Z, W, and X in 3 hours. Suppose the pipes all open, and the fountain full, in what time will it be emptied? *Ans.* 6$\frac{6}{19}$ hours.

CHAPTER XVIII.

ALLIGATION.

1. ***Alligation*** is the process employed in the solution of problems relating to the compounding of articles of different values or qualities.

2. ***Alligation*** { 1. Alligation Medial.
2. Alligation Alternate.

I. ALLIGATION MEDIAL.

1. ***Alligation Medial*** is the process of finding the mean, or average, rate of a mixture composed of articles of different values or qualities, the quantity and rate of each being given.

I. A grocer mixed 120 lb. of sugar at 5¢ a pound, 150 lb. at 6¢., and 130 lb. at 10¢.; what is the value of a pound of the mixture?

II. {
1. 120 lb. @5¢=$6.00,
2. 150 lb. @6¢=$9.00, and
3. *130* lb. @10¢=$*13.00*.
4. 400 lb. is worth $28.00.
5. ∴ 1 lb. is worth $28÷400=$.07=7 cents.

III. ∴ One pound of the mixture is worth 7 cents.

(*Stod. Comp. A., p. 244, prob. 3.*)

II. ALLIGATION ALTERNATE.

1. ***Alligation Alternate*** is the process of finding in what ratio, one to another, articles of different rates of quality or value must be taken to compose a mixture of a given mean, or average, rate of quality or value.

CASE I.

Given the value of several ingredients, to make a compound of a given value.

I. What relative quantities of tea, worth 25, 27, 30, 32, and 45 cents per lb. must be taken for a mixture worth 28 cents per lb.

		Diff.		*Bal.*		
28¢	25¢	3¢	2 lb.	17 lb.		19 lb.
	27¢	1¢			4 lb.	4 lb.
	30¢	2¢	3 lb.			3 lb.
	32¢	4¢			1 lb.	1 lb.
	45¢	17¢		3 lb.		3 lb.

SOLUTION. — In average, the principle is, that the gains and loses are equal. We write the average price and the particular values 25, 27, 30, 32, and 45 as in the margin. This is only a convenient

arrangement of the operation. Now one pound bought for 25¢ and sold in a mixture worth 28¢ there is a gain of 28¢—25¢, or 3¢; one pound bought at 27¢ and sold in a mixture worth 28¢, there is a gain of 28¢—27¢, or 1¢; one pound bought at 30¢ and sold in a mixture worth 28¢ there is a loss of 30¢—28¢, or 2¢; one pound bought at 32¢ and sold in a mixture worth 28¢, there is a loss of 32¢—28¢, or 4¢; and one pound bought at 45¢ and sold in a mixture worth 28¢ there is a loss of 45¢—28¢, or 17¢. Since the gains and losses are equal, we must take the ingredients composing this mixture in such a proportion as to make the gains and losses balance. We will first balance the 25¢ tea and the 30¢ tea. Since we gain 3¢ a pound on the 25¢ tea, and lose 2¢ on the 30¢ tea, how many pounds of each must we take so that the gain and loss on these two kinds may be equal? Evidently, we should gain 6¢ and lose 6¢. To find this, we simply find the L. C. M. of 3 and 2. Now if we gain 3¢ on one pound of the 25¢ tea, to gain 6¢, we must take as many pounds as 3¢ is contained in 6¢, which are 2 lb. If we lose 2¢ on one pound of the 30¢ tea, to lose 6¢, we must take as many pounds as 2¢ is contained in 6¢, which are 3 lb. Next, balance the 25-cent tea and the 45-cent tea. The L. C. M. of 3¢ and 17¢ is 51¢. Now if we gain 3¢ on one pound of the 25-cent tea to gain 51¢, we must take as many pounds as 3¢ is contained in 51¢ which are 17 lb. If we lose 17¢ on one pound of the 45-cent tea, to lose 51¢, we must take as many pounds as 17¢ is contained in 51¢ which are 3 lb. Next, balance the 27-cent tea and the 32-cent tea. The L. C. M. of 1¢ and 4¢ is 4¢. If we gain 1¢ on one pound of the 27-cent tea, to gain 4¢, we must take as many pounds as 1¢ is contained in 4¢, which are 4 lb. If we lose 4¢ on one pound of the 32-cent tea, it balances the gain on the 27-cent tea. Placing the number of pounds to be taken of each kind as shown above, and then adding horizontally, we have 19 lb. at 25¢, 4 lb. at 27¢, 3 lb. at 30¢, 1 lb. at 32¢, and 3 lb. at 45¢. It is not necessary to balance them in any particular order. All that must be observed, is that all the ingredients be used in balancing.

Note.—To prove the problem, use Alligation Medial.

CASE II.

To proportionate the parts, one or more of the quantities, but not the amount of the combination, being given.

I. How many bushels of hops, worth respectively 50, 60, and 75¢ per bushel, with 100 bushels at 40¢ per bushel, will make a mixture worth 65¢ a bushel?

$$A.\quad 65\text{¢}.\left|\begin{array}{l|l||l|l|l|l}
 & \textit{Dif.} & \textit{Bal.} & & & \\
40\text{¢} & 25\text{¢} & 2\text{ bu.} & & & 2\text{ bu.}\\
50\text{¢} & 15\text{¢} & & 2\text{ bu.} & & 2\text{ bu.}\\
60\text{¢} & 5\text{¢} & & & 2\text{ bu.} & 2\text{ bu.}\\
75\text{¢} & 10\text{¢} & 5\text{ bu.} & 3\text{ bu.} & 1\text{ bu.} & 9\text{ bu.}
\end{array}\right\}\times 50=\left\{\begin{array}{l}100\text{ bu.}\\100\text{ bu.}\\100\text{ bu.}\\450\text{ bu.}\end{array}\right.$$

$$B.\quad 65\text{¢}.\left|\begin{array}{l|l||l|l|l}
 & \textit{Dif.} & \textit{Bal.} & & \\
40\text{¢} & 25\text{¢} & 2\text{ bu.} & & \\
50\text{¢} & 15\text{¢} & & 2\text{ bu.} & \\
60\text{¢} & 5\text{¢} & & & 2\text{ bu.}\\
75\text{¢} & 10\text{¢} & 5\text{ bu.} & 3\text{ bu.} & 1\text{ bu.}
\end{array}\right\}\times 50=\left\{\begin{array}{r}100\text{ bu.}\\2\text{ bu.}\\2\text{ bu.}\\254\text{ bu.}\end{array}\right.$$

SOLUTION.—In this solution, we proceed as in Case I. In A, we obtain the relative amounts to be used of each kind, which is 2 bu. at 40¢, 2 bu. at 50¢, 2 bu. at 60¢, and 9 bu. at 75¢. But we are to have 100 bu. of the first kind. Hence, we must multiply these results by 100÷2, or 50. Doing this, we obtain 100 bu. at 40¢, 100 bu. at 50¢, 100 bu. at 60¢, and 450 bu. at 75¢.

Since either or both of the balancing columns, except the first, may be multiplied by any number whatever without affecting the average, it follows that there are an infinite number of results satisfying the conditions of the problem. Since we are to have 100 bu. at 40¢, the first column can be multiplied by only 50.

In B, we have multiplied the first column by 50 and added in the results in the other two columns. This gives us 100 bu. at 40¢, 2 bu. at 50¢, 2 bu. at 60¢, and 254 bu. at 75¢. The second and third columns may be multiplied by any number whatever. But the first must always must be multiplied by 50, because we are to have 100 bu. at 40 cents per bushel.

(*R. H. A., p. 338, prob. 2.*)

I. How much lead, specific gravity 11, with $\frac{1}{2}$ oz. copper, sp. gr. 9, can be put on 12 oz. of cork, sp. gr. $\frac{1}{4}$, so that the three will just float, that is, have a sp. gr. (1) the same as water?

$$1\left|\begin{array}{c|c||c|c}
\frac{1}{11} & \frac{10}{11} & 3 & \\
\frac{1}{9} & \frac{8}{9} & & 3\\
4 & 3 & \frac{10}{11} & \frac{8}{9}
\end{array}\right\}\times\frac{1}{6}\Big\}\times\frac{352}{27}=\left\{\begin{array}{l}39\frac{1}{9}\text{ oz.}=2\text{ lb. }7\frac{1}{9}\text{ oz.}\\ \frac{1}{2}\text{ oz.}\\ 12\text{ oz.}\end{array}\right.$$

SOLUTION.—The specific gravity of any body is the ratio which shows how many times heavier the body is than an equal

volume of water. Thus, when we say that the specific gravity of lead is 11, we mean that a cubic inch, a cubic foot, a cubic yard, or any quantity whatever is 11 times as heavy as an equal quantity of water.

Now if a cubic inch (say) of lead be immersed in water, it will displace a cubic inch of water; and since it weighs 11 times as much as a cubic inch of water, it displaces $\frac{1}{11}$ of its own weight. Hence, to have equal weights of water and lead we must take only $\frac{1}{11}$ as much lead as water. Now since a volume of water and $\frac{1}{11}$ as much lead have the same weight, and in the proper combination have a volume of 1, since the sp. gr. of the combination is 1, there is a loss of $1-\frac{1}{11}$, or $\frac{10}{11}$, in volume on the part of the lead. For the same reason, there is a loss of $\frac{8}{9}$ in volume on the part of the copper, and 3 on the part of the cork. Balancing, we see that we must take 3 volumes of lead with $\frac{10}{11}$ volumes of cork, a unit volume of water being the basis, in order that the two substances will just float, *i. e.*, have a specific gravity (1). In like manner, we must take 3 volumes of copper with $\frac{8}{9}$ volumes of cork. Now since we must always take 3 volumes of lead for every $\frac{10}{11}$ volumes of cork, it is evident that the weights of the substances are in the same proportion. Hence, we may say, we must take 3 oz. of lead with every $\frac{10}{11}$ oz. of cork, and 3 oz. of copper with every $\frac{8}{9}$ oz. of cork.

But we are to have only $\frac{1}{2}$ oz. of copper. Hence, we must multiply the second balancing column by some number that will give us $\frac{1}{2}$ oz. of copper, *i. e.*, we must multiply 3 by some number that will give us $\frac{1}{2}$. The number by which we must multiply is $\frac{1}{2}\div3=\frac{1}{6}$. But multiplying $\frac{8}{9}$ by $\frac{1}{6}$, we get $\frac{4}{27}$ oz. of cork. But we are to have altogether 12 oz. of cork. Hence we must yet have 12 oz.$-\frac{4}{27}$ oz.$=\frac{320}{27}$ oz. To produce this, we must multiply $\frac{10}{11}$ by some number that will give $\frac{320}{27}$ oz. This number is $\frac{320}{27}\div\frac{10}{11}=\frac{352}{27}$. But we must also multiply 3 by $\frac{320}{27}$. This will give us $39\frac{1}{9}$ oz.=2 lb. $7\frac{1}{9}$ oz. of lead. Hence, we must use 2 lb. $7\frac{1}{9}$ oz. of lead, so that the three will just float.

(*R. H. A., p. 339, prob. 7.*)

I. How many shares of stock, at 40%, must A buy, who has bought 120 shares, at 74%, 150 shares, at 68%, and 130 shares, at 54%, so that he may sell the whole at 60%, and gain 20%?

1. (1.) 100%=the average cost.
(2.) 20%=gain.
(3.) 120%=the average selling price.
(4.) 60%=the average selling price.
(5.) ∴ 120%=60%.
(6.) 1%=$\frac{1}{120}$ of 60%=$\frac{1}{2}$%.
(7.) 100%=100 times $\frac{1}{2}$%=50%, the average cost.

II. 2. (1.) 120 shares @ 74% = 8880%.
(2.) 150 shares @ 68% = 10200%.
(3.) 130 shares @ 54% = 7020%.
(4.) ∴ 400 shares are worth 26100%, and
(5.) 1 share is worth 26100% ÷ 400 = $65\frac{1}{4}$%, the average.

$$3.\quad 50\%\begin{array}{|l|l||l|} 40\ \% & 10\ \% & 15\frac{1}{4}\text{ shares.} \\ & & \\ 65\frac{1}{4}\% & 15\frac{1}{4}\% & 10\text{ shares.}\end{array}\Big\}\times 40=\begin{cases}610\text{ shares.}\\ 400\text{ shares.}\end{cases}$$

III. ∴ He must take 610 shares. (*R. H. A., p. 339, prob. 8.*)

Explanation.—Since 60% is the average selling price, and his gain is 20%, it is evident that his average cost is 60% ÷ 1.20, or 50%. In step 3, we find that the average cost of the 400 shares is $65\frac{1}{4}$%. Hence, the problem is the same as to find how many shares at 40%, must A buy who has 400 shares at at an average of $65\frac{1}{4}$% so that his average cost will be 50%. Balancing, we find that he must take $15\frac{1}{4}$ shares at 40% with 10 shares at $65\frac{1}{4}$%. But he has 400 shares at $65\frac{1}{4}$%. Hence, we must multiply the balancing column by 400 ÷ 10, or 40. This gives 610 shares at 40%.

CASE III.

To proportion the parts, the amount of the whole combination being given.

I. How many barrels of flour, at \$8, and \$8.50, with 300 bbl. at \$7.50, 800 bbl. at \$7.80, and 400 bbl. at \$7.65, will make 2000 bbl. at \$7.85 a bbl.?

II. 1. 300 bbl. @ \$7.50 a bbl. = \$2250.
2. 800 bbl. @ \$7.80 a bbl. = \$6240.
3. 400 bbl. @ \$7.65 a bbl. = \$3060.
4. ∴ 1500 bbl. are worth \$11550.
5. \$7.85 = the average price per bbl. of 2000 bbl.
6. ∴ \$15700 = 2000 × \$7.85 = the value of 2000 bbl.
7. ∴ \$15700 — \$11550 = \$4150 = the value of 2000 bbl. — 1500 bbl., or 500 bbl.
8. ∴ \$8.30 = \$4150 ÷ 500 = the average value of 1 bbl.

$$9.\quad \$8.30\begin{array}{|l|l|l|} \$8.00 & \$.30 & 2\text{ bbl.} \\ \hline \$8.50 & \$.20 & 3\text{ bbl.} \\ & & \overline{5\text{ bbl.}}\end{array}\Big\}\times(500\div 5)=\begin{cases}200\text{ bbl.}\\ 300\text{ bbl.}\end{cases}$$

III. ∴ 1. 200 bbl. at \$8.00 per bbl. must be taken with
2. 300 bbl. at \$8.50 per bbl.

(*R. H. A., p. 339, prob. 2.*)

I. A dealer in stock can buy 100 animals for \$400, at the following rates: calves, \$9; hogs, \$2; lambs, \$1; how many may he take of each kind?

			Bal.										
	$1	$3	5 lambs.		3	10	17	24	31	38	45	62	59
$4	$2	$2		5 hogs.	68	60	52	44	36	28	20	12	4
	$9	$5	3 calves.	2 calves.	29	30	31	32	33	34	35	36	37
			8	7									

Explanation.—A lamb bought for $1 and sold for $4 is a gain of $3; a hog bought for $2 and sold for $4 is a gain of $2; and a calf bought for $9 and sold for $4 is a loss of $5. We must make the gains and loses equal. The L. C. M. of $3 and $5 is $15. If we gain $3 on one lamb to gain $15 we must take as many lambs as $3 is contained in $15, which are 5 lambs. If we lose $5 on one calf, to lose $15, we must take as many calves as $5 is contained in $15, which are 3 calves. The L. C. M. of $2 and $5 is $10. If we gain $2 on one hog, to gain $10, we must take as many hogs as $2 is contained in $10, which are 5 hogs. If we lose $5 on one calf, to lose $10, we must take as many calves as $5 is contained in $10, which are 2 calves. Adding the balancing columns, considering them as abstract numbers, we have 8 and 7. $8+7=15$. $100\div15=6\frac{2}{3}$. ∴ Multiplying each balancing column by $6\frac{2}{3}$, will give $33\frac{1}{3}$ lambs, $33\frac{1}{3}$ hogs, and $33\frac{1}{3}$ calves. But this result is not compatible with the nature of the problem. Hence we must see if we can take a number of 8's and a number of 7's that will make 100. By trial, we find that *two* 8's and *twelve* 7's will make 100. Hence, multiplying the first column by 2 and the second by 12, and adding the columns horizontally, we have for our result, 10 lambs, 60 hogs, and 30 calves. Again, we find, by trying three 8's, four 8's, and so on, that *nine* 8's taken from 100, will leave 28 which is *four* 7's. Hence, nine 8's and four 7's will make 100. Then, multiplying the first column by 9 and the second by 4, and adding the columns horizontally, we have for a second result 45 lambs, 20 hogs, and 35 calves. Now these are the only answers that can be obtained by taking an integral number of 8's and integral number of 7's to make 100. But other answere may be obtained by taking 8 a *fractional* number of times, and 7 a fractional number of times to make 100. Suppose, for illustration, we try to take a number of thirds 8 times. We find that 8 taken 6-third times and 7 taken 36 third times will make 100. Multiplying the first column by $\frac{6}{3}$ and the second by $\frac{36}{3}$, and adding the columns horizontally, we have, for a result, 10 lambs, 60 hogs, and 30 calves—the same as that obtained by taking 8 twice and 7 twelve times. Again, we find, that 8 taken 13 third times and 7 taken 28-third times will make 100. Multiplying and adding as before we find that our results are fractional. Hence, we can not take a fraction whose denominator is three. It is clear that we must take a fractlon whose denominator will reduce to unity when being multiplied by 5. Hence, if we try to take 8 a number of fifths times and 7 a number of fifths times to make 100, our results will all be integral. By trial, we find that 8 taken 3-fifths times and 7 taken 68-fifths times will make 100. Multiplying and adding as before, we have, for our results, 3 lambs, 68 hogs, and 29 calves. Again, we find that 8 taken 10-fifths times and 7 taken 60-fifths times, will make 100. Multiplying and adding as before, we have, for results, 10 lambs, 60 hogs, and 30 calves. Again, by trial, we find that 8 taken 17-fifths and 7 taken 52-fifths times will make 100. Multiplying the first column by $\frac{17}{5}$ and the second by $\frac{52}{5}$, and adding the columns horizontally, we have, for results, 17 lambs, 52 hogs, and 31 calves. Continuing the process, we find nine admissible answers. These are the only answers, satisfying the nature of the problem.

CHAPTER XIX.

SYSTEMS OF NOTATION.

1. ***A System of Notation*** is a method of expressing numbers by means of a series of powers of some fixed number called the *Radix*, or *Base* of the scale in which the different numbers are expressed.

2. ***The Radix*** of any system is the number of units of one order which makes one of the next higher.

3.

Names of Systems.	Radix.	Names of Systems.	Radix.
Unitary - -	1	Nonary - -	9
Binary - -	2	Decimal, or Denary	10
Ternary - -	3	Undenary -	11
Quaternary -	4	Duodenary, -	12
Quinary - -	5	Vigesimal, -	20
Senary - -	6	Trigesimal, -	30
Septenary - -	7	Sexagesimal, -	60
Octonary - -	8	Centesimal, -	100

4. In writing any number in a uniform scale, as many distinct characters, or symbols, are required as there are units in the radix of the given system. Thus, in the decimal system, 10 characters are required; in the ternary, 3; viz., 1, 2, and 0; in the senary, 6; viz., 1, 2, 3, 4, 5, and 0; and so on.

5. Let r be the radix of any system, then any number, N, may be expressed in the form,

$N=ar^n+br^{n-1}+cr^{n-2}+dr^{n-3}+\ldots\ldots+pr^2+qr+s$, in which the co-efficients a, b, c, ... , are each less than r.

To express an integral number in a proposed scale: *Divide the number by the radix, then the quotient by the radix, and so on; the successive remainders taken in order will be the successive digits beginning from units place.*

I. Express the common number, 75432, in the senary system.

II.
1. 6)75432
2. 6)12572+0
3. 6)2095+2
4. 6)349+1
5. 6)58+1
6. 6)9+4
7. 1+3

III. ∴ 75432 in the decimal system=1341120 expressed in the senary system.

I. Transform 3256 from a scale whose radix is 7, to a scale whose radix is 12.

II.
1. 12)3256
2. 12)166+4
3. 12)11+1
4. 0+8

III. ∴ 3256 in the septenary system=814 in the duodenary system.

Explanation.—In the senary system, 7 units of one order make one of the next higher. Hence, 3 units of the fourth order = 7×3, or 21, units of the third order. 21 units +2 units = 23 units. 23÷12 = 1, with a remainder 11. 11 units of the third order = 77 units of the second order. 77 units +5 units=82 units. 82÷12=6, with a remainder 10. 10 units of the second order=70 units of the first order. 70 units +6 units=76 units. 76 ÷12=6, with a remainder 4. Hence, the first quotient is 166, with a remainder 4. Treat this quotient in like manner, and so on, until a quotient is obtained, that is less than 12.

I. What is the sum of 45324502 and 25405534, in the senary system?

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 45324502
 25405534
115134440
```

Explanation.—4+2=6. 6÷6=1, with no remainder. Write the 0 and carry the 1. 3+1=4. Write the 4. 5+5=10. 10÷6=1, with a remainder 4. Write the 4 and carry the 1. 5+4+1=10. 10÷6=1, with a remainder 4. Write the 4 and carry the 1. 0+2+1=3. Write the 3. 4+3=7. 7÷6+1 with a remainder 1. Write 1 and carry 1 5+5+1=11. 11÷6= 1, with a remainder 5. Write the 5 and carry the 1. 2+4+1=7. 7÷6= 1, with a remainder 1. Write 1 and carry 1. The result is 115134440.

I. What is the difference between 24502 and 5534 in the octonary system?

```
24502
 5534
16746
```

Explanation.—4 cannot be taken from 2. Hence, borrow one unit from a higher denomination. Then (2+8)—6=4. (8—1)—3=4. 5 from (4+8) =7. 5 from (3+8)=6. Hence, the result is 16746.

I. Transform 3413 from the scale of 6 to the scale of 7.

$$\text{II.}\begin{cases} 1. & 7\underline{)3413} \\ 2. & 7\underline{)310+3} \\ 3. & 7\underline{)24+3} \\ & 2+2 \end{cases}$$

III. $\therefore$ 3413 in the senary system=2233 in the septenary system.

I. Multiply 24305 by 34120 in the senary system.

$$\begin{array}{r} 24305 \\ 34120 \\ \hline 530140 \\ 24305 \\ 150032 \\ 121323 \\ \hline 1411103040 \end{array}$$

Explanation.—Multiplying 5 by 2 gives 10. 10÷6=1, with a remainder 4. Write 4 and carry 1 to the next order. 2 times 0=0. 0+1=1. Write the 1. 2 times 3=6. 6÷6=1, with a remainder 0. Write the 0 and carry the 1 to the next higher order. 2 times 4=8. 8+1=9. 9÷6=1, with a remainder 3. Write 3 and carry the 1 to the next higher order. 2 times 2=4. 4+1=5. Write 5. Multiply in like manner by 1, 4, and 3. Add the partial products, remembering that 6 units of one order, in the senary system, uniformly make one of the next higher.

I. Multiply 2483 by 589 in the undenary system, or system whose radix is 11.

We must represent 10 by some character. Let it be *t*.

$$\begin{array}{r} 2483 \\ 589 \\ \hline 1t985 \\ 1t502 \\ 11184 \\ \hline 13322t5 \end{array}$$

Explanation.—In the undenary system, 11 units of one order uniformly make one of the next higher order. 9 times 3=27. 27÷11=2, with a remainder 5. Write 5 and carry the 2 to the next higher order, or second order. 9 times 8=72. 72+2=74. 74÷11=6, with a remainder 8. Write 8 and carry the 6 to the next higher order, or third order. 9 times 4=36. 36+6=42. 42÷11=3, with a remainder 9. Write 9 and carry the 3 to the next higher order, or the fourth order. 9 times 2=18. 18+3=21. 21÷11 =1, with a remainder *t*. Write *t* and carry the 1 to the next higher order. Multiply in like manner by 8 and 5. Add the partial products, remembering that 11 units of one order equals one of the next higher. Wherever 10 occurs, it must be represented by a single character *t*.

I. Divide 1184323 by 589 in the duodenary system.

In the duodenary system, we must have 12 characters ; viz., 1, 2, 3, 4, 5, 6, 7, 8, 9, *t*, *e*, and 0. *t* represents 10 and *e*, 11.

```
589)1184323(2486
    e56
    ----
    22t3
    1te0
    -----
     3e32
     39t0
     -----
      1523
      1523
```

Explanation.—In the duodenary system, 12 units of one order make one of the next higher. 1184 will contain 589, 2 times. Then multiply the divisor, 589, by 2 thus: 2 times 9=18. 18÷12=1, with a remainder 6. Write the 6 and carry the 1. 2 times 8=16. 16+1=17. 17÷12=1, with a remainder 5. Write the 5 and carry the 1. 2 times 5=*t*. *t*+1=*e*. Write the *e*. Then subtract. 6 from (12+4)=*t*, 5 from 7=2, and *e* from (12+1)=2.

Hence, the first partial dividend is 22 *t*. Bring down 3. Then 22t3 will contain 589, 4 times. Multiply as before. By continuing the operation we obtain 2483 for a quotient.

I. Divide 95088918 by *tt*4, in the duodenary system.

```
tt4)95088918(t4tee
    9074
    ----
     4548
     3754
     ----
      9e49
      9074
      ----
       t951
       9e58
       ----
        te58
        te58
```

I. Extract the square root of 11122441 in the senary system.

```
               11122441(2405
   2×2=         4
           44 | 312
              | 304
   2×24=52   0|42441
   2×240=520 5|42441
```

Explanation.—The greatest square in 11 expressed in the senary system is 4. Subtracting and bringing down the next period, we have 312 for the next partial dividend. Doubling the root already found and finding how many times it is contained in 312 expressed in the senary system, we find it is 4. Continuing the process the same as in the decimal system, the result is 2405.

I. Extract the square root of 11000000100001 in the binary system.

```
                 11000000100001(1101111
                 1
        101     |1000
                | 101
                 ------
                 |110000
      11001      | 11001
                  -------
                  |1011110
     110101       | 110101
                   --------
                   |10100100
    1101101        | 1101101
                    ---------
                    |11011101
   11011101         |11011101
```

(*Todhunter's Algebra, p. 255, Ex. 23.*)

I. Find in what scale, or system, 95 is denoted by 137.

II.
1. Let r=the radix of the system. Then
2. $r^2+3r+7=95$,
3. $r^2+3r=95-7=88$, and
4. $r^2+3r+\frac{9}{4}=88+\frac{9}{4}=\frac{361}{4}$, by completing the square.
5. $r+\frac{3}{2}=\frac{19}{2}$, by extracting the square root, and
6. $r=\frac{19}{2}-\frac{3}{2}=\frac{16}{2}=8$, the radix of the system.

III. ∴ 95 is denoted by 137 in the octonary system.

(*Todhunter's Alg., p. 255, prob. 26.*)

I. Find in what system 1331 is denoted by 1000.

II.
1. Let r=the radix of the system. Then
2. $r^4+0r^3+0r^2+0r+0=1331$, or
3. $r^4=1331$. Whence
4. $r=\sqrt[4]{1331}=11$, the radix of the system.

III. ∴ 1331 is denoted by 1000 in the undenary system.

(*Todhunter's Alg., p. 255, prob. 28.*)

CHAPTER XX.

MENSURATION.

1. ***Mensuration*** is that branch of applied mathematics which treats of *geometrical magnitudes.*

2. ***Geometrical Magnitudes*** are *lines*, *surfaces*, and *solids.*

3. *Geometrical Magnitudes.*

- A. Line.
  - 1. Straight.
  - 2. Curved.
- B. Surface.
  - 1. Plane.
    - 1^1. Polygon.
      - 1^2. Triangle.
        - 1^3. Right.
          - 1^4. Isosceles.
          - 2^4. Scalene.
        - 2^3. Oblique.
          - 1^4. Isosceles.
          - 2^4. Scalene.
          - 3^4. Equilateral.
      - 2^2. Quadrilateral.
        - 1^3. Parallelogram.
          - 1^4. Right ‖ Rectangle ‖ Square.
          - 2^4. Oblique ‖ Rhomboid ‖ Rhombus.
        - 2^3. Trapezoid.
        - 3^3. Trapezium.
      - 3^2. Pentagon.
      - 4^2. &c.
    - 2^1. Circle.
    - 3^1. Ellipse.
    - 4^1. &c.
  - 2. Curved.
    - 1^1. Spherical Surface.
    - 2^1. Conical Surface.
    - 3^1. Cylindrical Surface.
    - 4^1. &c.
- C. Solid.
  - 1. Polyhedron.
    - 1^1. Prism.
      - 1^2. As to No. of sides in the base.
        - 1^3. Triangular.
        - 2^3. Quadrangular.
        - 3^3. Pentagonal.
        - 4^3. Hexagonal.
        - 5^3. &c.
      - 2^2. Parallelopipedon.
        - 1^3. Right | Rectangular | Cube.
        - 2^3. Oblique.
    - 2^1. Pyramid.
      - 1^2. As to No. of sides in the base.
        - 1^3. Triangular.
        - 2^3. Quadrangular.
        - 3^3. &c.
      - 2^2. Right.
      - 3^2. Oblique.
      - 4^2. Tetrahedron.
    - 3^1. Octahedron.
    - 4^1. Dodecahedron.
    - 5^1. Icosahedron.
  - 2. Cylinder.
    - 1^1. Right.
    - 2^1. Oblique.
  - 3. Cone.
    - 1^1. Right.
    - 2^1. Oblique.
  - 4. Sphere.
  - 5. &c.

4. *A Line* is a geometrical magnitude having length, without breadth or thickness.

5. *A Straight Line* is a line which pierces space evenly, so that a piece of space from along one side of it will fit any side of any other portion.

6. *A Curved Line* is a line no part of which is straight.

7. *A Surface* is the common boundary of two parts of a solid, or of a solid and the remainder of space.

8. *A Plane Surface,* or ***Plane,*** is a surface which divides space evenly, so that a piece of space from along one side of it will fit either side of any other portion of it.

9. *A Curved Surface* is a surface no part of which is plane.

10. *A Polygon* (*Πολυγωνος*, from *Πολυς*, many, and *γωνια*, angle) is a portion of a plane bounded by straight lines.

11. *A Circle* (*κιρκος*, circle, ring) is a portion of a plane bounded by a curved line every point of which is equally distant from a point within called the center,

12. *An Ellipse* (*ἐλλειψις*) is a portion of a plane bounded by a curved line any point from which, if two straight lines are drawn to two points within, called the *foci*, the sum of the two lines will be constant.

13. *A Triangle* (Lat. *Triangulum*, from *tries*, *tria*, three, and *angulus*, corner, angle) is a polygon bounded by three straight lines.

14. *An Angle* is the opening between two lines which meet in a point.

15. *Angles*
- 1. Straight Angle.
- 2. Right Angle.
- 3. Oblique
  - 1. Acute.
  - 2. Obtuse.

16. *A Straight Angle* has its sides in the same line, and on different sides of the point of meeting, or *vertex*.

17. *A Right Angle* is half of a *Strait Angle*, and is formed by one straight line meeting another so as to make the adjacent angles equal.

18. *An Oblique Angle* is formed by one line meeting another so as to make the adjacent angles unequal.

19. *An Acute Angle* is an angle less than a right angle.

20. *An Obtuse Angle* is an angle greater than a right angle.

21. ***A Right Triangle*** is a triangle, one of whose angles is a right angle.

22. ***An Oblique-Angled Triangle*** is one whose angles are all oblique.

23. ***An Isosceles Triangle*** is one which has two of its sides equal.

24. ***A Scalene Triangle*** is one which has no two of its sides equal.

25. ***An Equilateral Triangle*** is one which has all the sides equal.

26. ***A Quadrilateral*** (Lat. *quadrilaterus*, from *quatuor*, four, and *latus*, *lateris*, a side) is a polygon bounded by four straight lines.

27. ***A Parallelogram*** (*Παραλληλόγραμμον*, from *Παραλληλος*, parallel, and *γραμμή*, a stroke in writing, a line) is a quadrilateral having its opposite sides parallel, two and two.

28. ***A Right Parallelogram*** is a parallelogram whose angles are all right angles.

29. ***An Oblique Parallelogram*** is a parallelogram whose angles are oblique.

30. ***A Rectangle*** (Lat. *rectus*, right, and *angulus*, an angle) is a right parallelogram.

31. ***A Square*** is an equilateral rectangle.

32. ***A Rhomboid*** (*ῥομβοειδης*, from *ῥόμβος*, rhomb, and *εἶδος*, shape) is a parallelogram whose angles are oblique.

33. ***A Rhombus*** (*ρομβος*, from *ῥέμβειν*, to turn or whirl round) is an equilateral rhomboid.

34. ***A Pentagon*** (*Πεντάγωνον*, *Πεντε*, five, and *γωνια*, angle) is a polygon bounded by five sides. Polygons are named in reference to the number of sides that bound them. A *Hexagon* has six sides; *Heptagon*, seven; *Octagon*, eight; *Nonagon*, nine; *Decagon*, ten; *Undecagon*, eleven; *Dodecagon*, twelve; *Tridecagon*, thirteen; *Tetradecagon*, fourteen; *Pentedecagon*, fifteen; *Hexdecagon*, sixteen; *Heptadecagon*, seventeen; *Octadecagon*, eighteen; *Enneadecagon*, nineteen; *Icosagon*, twenty; *Icosaisagon*, twenty-one; *Icosadoagon*, twenty-two; *Icosatriagon*, twenty-three; *Icosatetragon*, twenty-four; *Icosapentegon*, twenty-five; *Icosahexagon*, twenty-six; *Icosaheptagon*, twenty-seven; *Icosaoctagon*, twenty-eight; *Icosaenneagon*, twenty-nine; *Triacontagon*, thirty; *Tricontaisagon*, thirty-one; *Tricontadoagon*, thirty-two; *Tricontatriagon*, thirty-three; and so on to *Tessaracontagon*, forty; *Pentecontagon*, fifty; *Hexacontagon*, sixty;

Hebdomacontagon, seventy; *Ogdoacontagon*, eighty; *Encnacontagon*, ninety; *Hecatonagon*, one hundred; *Diacosiagon*, two hundred; *Triacosiagon*, three hundred; *Tetracosiagon*, four hundred; *Pentecosiagon*, five hundred; *Hexacosiagon*, six hundred; *Heptacosiagon*, seven hundred; *Oktacosiagon*, eight hundred; *Enacosiagon*, nine hundred; *Chiliagon*, one thousand; &c.

35. ***A Spherical Surface*** is the boundary between a sphere and outer space.

36. ***A Conical Surface*** is the boundary between a cone and outer space.

37. ***A Cylindrical Surface*** is the boundary between the cylinder and outer space.

38. ***A Solid*** is a part of space occupied by a physical body, or marked out in any other way.

39. ***A Polyhedron*** (*Πολυεδρος*, from *Πολυς*, many, and *ἕδρα*, seat, base) is a solid bounded by polygons.

40. ***A Prism*** is a polyhedron in which two of the faces are polygons equal in all respects and having their homologous sides parallel.

41. ***The Altitude*** of a prism is the perpendicular distance between the planes of its bases.

42. ***A Triangular Prism*** is one whose bases are triangles.

43. ***A Quadrangular Prism*** is one whose bases are quadrilaterals.

44. ***A Parallelopipedon*** is a prism whose bases are parallelograms.

45. ***A Right Parallelopipedon*** is one whose lateral edges are perpendicular to the planes of the bases.

46. ***A Rectangular Parallelopipedon*** is one whose faces are all rectangles.

47. ***A Cube*** (*κυβος*, a cube, a cubical die) is a rectangular parallelopipedon whose faces are squares.

48. ***A Right Prism*** is one whose lateral edges are perpendicular to the planes of the bases.

49. ***An Oblique Prism*** is one whose lateral edges are oblique to the planes of the bases.

50. ***A Pyramid*** (*Πυραμίδος*) is a polyhedron bounded by a polygon called the *base*, and by triangles meeting at a common point called the *vertex* of the pyramid.

51. ***The Convex Surface*** of a pyramid is the sum of the triangles which bound it.

52. ***A Right Pyramid*** is one whose base is a regular polygon, and in which the perpendicular, drawn from the vertex to the plane of the base, passes through the center of the base. The perpendicular is called the *axis*.

53. ***A Tetrahedron*** (τέτρα, four, and ἕδρα, seat, base) is a pyramid whose faces are all equilateral triangles.

54. ***The Altitude*** (Lat. *Altitudo*, from *altus*, high, and *ude* denoting state or condition) of a pyramid is the perpendicular distance from the vertex to the plane of the base.

55. ***The Slant Height*** of a pyramid, is the perpendicular distance from the vertex to any side of the base.

56. ***A Triangular Pyramid*** is one whose base is a triangle.

57. ***An Octahedron*** (οκταεδρος, from οκτα, eight, and ἕδρα, seat, base) is a polyhedron bounded by eight equal equilateral triangles.

58. ***A Dodecahedron*** (δώδεκα, twelve, and ἕδρα, *seat, base*) is a polyhedron bounded by twelve equal and regular pentagons.

59. ***An Icosahedron*** (εἴκοσί, twenty, and ἕδρα, seat, base) is a polyhedron bounded by twenty equal equilateral triangles.

60. ***A Cylinder*** (κυλινδρος, from, κυλινδρειν, κυλειν, to roll) is a solid bounded by a surface generated by a line so moving that every two of its positions are parallel, and two parallel planes.

61. ***The Axis*** (αξων) of a cylinder is the line joining the centers of its bases.

62. ***A Right Cylinder*** is one whose axis is perpendicular to the planes of the bases.

63. ***A Cone*** (κωνος, from Skr. ço, to bring to a point) is a solid bounded by a surface generated by a straight line moving so as always to pass through a fixed point called the *apex*,, and a plane.

64. ***A Right Cone*** is a solid generated by revolving a right-angled triangle about one perpendicular.

65. ***An Oblique Cone*** is one in which the line, called the *axis*, drawn from the apex to the center of the base is not perpendicular.

66. ***The Frustum*** (Lat. *frustum*, piece, bit) of a pyramid or a cone is the portion included between the base and a parallel section.

67. ***A Sphere*** (σφαῖρα) is a solid bounded by a curved

surface, every point of which is equally distant from a point within, called the center.

Before we enter into the solution of problems in Mensuration, it will be necessary first to explain a difficulty which we encounter.

The common way of teaching that *feet* multiplied by *feet* give *square feet* is wrong; for there is no rule in mathematics justifying the multiplication of one denominate number by another. If it is correct to say *feet* multiplied by *feet* give *square feet*, we might, with equal propriety, say *dollars* multiplied by *dollars* give *square dollars*—a product wholly unintelligible. In all our reasoning, we deal with abstract numbers alone or the symbols of abstract numbers. These do not represent lines, surfaces, or solids, but the relations between these numbers may represent the relations between the magnitudes under consideration.

Suppose, for example, that the line AB contains 5 units, and the the line BC 4 units. Let a denote the abstract number 5, and b the abstract number 4. Then $ab=20$. Now this product ab is not a surface, nor the representation of a surface. It is simply the abstract number 20. But this number is exactly the same as the number of square units contained in the rectangle whose sides are AB and BC, as may be seen by constructing the rectangle $ABCD$. Hence the surface of the rectangle is measured by 20 squares described on the unit of length.

This relation is universal, and we may always pass from the abstract thus obtained by the product of any two letters, to the measure of the corresponding rectangle by simply considering the abstract units as so many concrete or denominate units.

In like manner, the product of three letters abc is not a solid obtained by multiplying lines together, which is an impossible operation. It is simply the product of three abstract numbers represented by the letters a, b, and c, and is consequently an abstract number. But this number contains precisely as many units as there are solid units in the parallelopipedon whose edges correspond to the lines a, b, and c; hence, we may easily pass from the abstract to the concrete. Hence, if we wish to find the area of a rectangle whose width is 4 feet and length 6 feet, we simply say, $6\times4=24$ square feet. We pass at once from the abstract in the first member to the concrete in the second.

It is a question whether pupils should be taught a falsehood in order that they may learn a truth.

(*See Bledsoe's Philosophy of Mathematics, pp. 97-106.*)

I. PARALLELOGRAMS.

Prob. I. To find the area of a parallelogram; whether it be a square, a rectangle, a rhomboid, or a rhombus.

Formula.— $A=l\times b$, where A=area, l=length, and b=breadth; or, $A=b\times a$, where A=area, b=base, and a=altitude.

Rule—*Multiply the length by the breadth; or, the base by the altitude.*

I. What is the area of a parallelogram whose length is 15 feet and breadth 7 feet?

By formula, $A=l\times b$=length$\times$breadth$=15\times7=105$ sq. feet.

II. $\begin{cases} 1.\ 15 \text{ feet=length.} \\ 2.\ 7 \text{ feet=breadth.} \\ 3.\ \therefore 15\times7=105 \text{ sq. ft.} \\ \quad =\text{area.} \end{cases}$

III. ∴ The area is 105 sq. ft.

FIG. 4.

Note.—The *base* is not necessarily the side toward the ground. Thus in the parallelogram $ABCD$, BC may be considered the base, in which case, the altitude would be the perpendicular distance EF, between the sides BC and AD. If HG and BC were given, we could not find the area of the parallelogram because we have not the base and altitude given.

I. What is the area of the parallelogram $ABCD$, if BC is 26 feet and EF 50 feet?

By formula, $A=a\times b=EF\times BC=50\times26=1300$ sq. ft.

II. $\begin{cases} 1.\ 26 \text{ feet}=BC=\text{base.} \\ 2.\ 50 \text{ feet}=EF=\text{altitude.} \\ 3.\ \therefore 26\times50=1300 \text{ sq. ft.=area.} \end{cases}$

III. ∴ The area of $ABCD$=1300 sq. ft.

I. A floor containing 132 square feet, is 11 feet wide; what is its length?

By formula, $A=l\times b$. $\therefore l=A\div b=132\div11=12$ ft.

II. $\begin{cases} 1.\ 132 \text{ sq. ft.=area.} \\ 2.\ 11 \text{ ft.=breadth.} \\ 3.\ 132\div11=12 \text{ ft.=length.} \end{cases}$

III. ∴ The floor is 12 ft. long.

Prob. II. The diagonal of a square being given, to find the area.

Formula.—$A=d^2\div2$.

Rule.—*Divide the square of the diagonal by 2, and the quotient will be the area.*

I. What is the area of a square whose diagonal is 8 chains?

By formula, $A=d^2\div2=8^2\div2=32$ sq. chains.

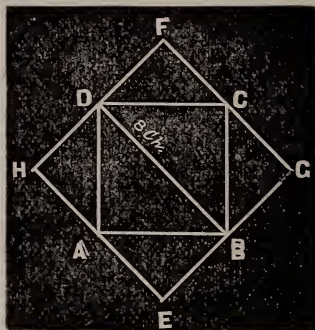

FIG. 5.

II.
1. 8 ch.=length of diagonal=BD.
2. 64 sq. ch.$=8\times8=EGFH=$ square described on the diagonal BD.
3. 32 sq. ch.=64 sq. ch.$\div2=$ area of the square $ABCD$.

III. ∴ 32 sq. ch.=the area of the square.

Prob. III. The area of a square being given, to find its diagonal.

Formula.—$d=\sqrt{2A}$.

Rule.—*Extract the square root of double the area.*

I. The area of a square is 578 sq. ft.; what is the diagonal?

By formula, $d=\sqrt{2A}=\sqrt{2\times578}=\sqrt{1156}=34$ feet.

II.
1. 578 sq. ft.=area of the square.
2. 1156 sq. ft.$=2\times578$ sq. ft.=double the area.
3. 34 feet$=\sqrt{1156}=$the diagonal.

III. ∴ The diagonal is 34 feet.

Prob. IV. The diagonal of a square being given, to find its side.

Formula.—$S=\sqrt{\frac{1}{2}d^2}$.

Rule.—*Extract the square root of one-half the square of the diagonal.*

I. What is the side of a square whose diagonal is 12 feet?

By formula, $S=\sqrt{\frac{1}{2}d^2}=\sqrt{\frac{1}{2}\times12^2}=\sqrt{72}=6\sqrt{2}=8.4852+$ft.

II.
1. 12 ft.=the diagonal.
2. 144 sq. ft.$=12^2=$square described on the diagonal.
3. 72 sq. ft.=area of square whose side is required.
4. ∴ 8.4852 ft.$=6\sqrt{2}=\sqrt{72}=$side of the square.

III. ∴ The side of the square is 8.4852+ft.

Prob. V. To find the side of a square having its area given.

Formula.—$S=\sqrt{A}$.

Rule.—*Extract the square root of the number denoting its area.*

I. What is the side of a square field whose area is 2500 square rods?

By formula, $S=\sqrt{A}=\sqrt{2500}=50$ rods.

II. $\begin{cases} \text{1. 2500 sq. rd.=area of the field.} \\ \text{2. 50 rd.}=\sqrt{2500}=\text{side of the square field.} \end{cases}$

III. ∴ The side of the field is 50 rods.

II. TRIANGLES.

Prob. VI. Given the base and altitude of a right-angled triangle, to find the hypothenuse.

Formula.—$h=\sqrt{a^2+b^2}$.

Rule.—*To the square of the base add the square of the altitude and extract the square root of the sum.*

I. In the right angled triangle ACB, the base $AC=56$ and the altitude $BC=33$; what is the hypothenuse?

By formula, $h=\sqrt{a^2+b^2}=\sqrt{33^2+56^2}=\sqrt{1089+3136}=\sqrt{4225}$ $=65$.

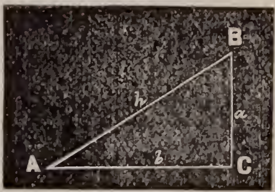

FIG. 6.

II.
1. $56=AC=$the base.
2. $3136=56^2=$ the square of the base.
3. $33=BC=$the altitude.
4. $1089=33^2=$the square of the altitude.
5. $4225=3136+1089=$the sum of the squares of the base and altitude.
6. $65=\sqrt{4225}=$the square root of the sum of the squares of the base and altitude=the hypothenuse.

III. ∴ The hypothenuse=65.

Prob. VII. To find a side, when the hypothenuse and the other side are given.

Formulas.—$\begin{cases} a=\sqrt{h^2-b^2}. \\ b=\sqrt{h^2-a^2}. \end{cases}$

Rule.—*From the square of the hypothenuse subtract the square of the given side and extract the square root of the remainder.*

I. The hypothenuse of a right-angled triangle is 109, and the altitude 60; what is the base?

By formula, $b=\sqrt{h^2-a^2}=\sqrt{109^2-60^2}=\sqrt{8281}=91$.

II.
1. 109=hypothenuse.
2. $11881=109^2$=square of the hypothenuse.
3. 60=the altitude.
4. $3600=60^2$=the square of the altitude.
5. 8281=11881—3600=difference of the squares of the hypothenuse and altitude.
6. $91=\sqrt{8281}$=the square root of this difference=the base.

III. ∴ The base is 91.

Remark.—When $a=b$, $h=\sqrt{2a^2}=a\sqrt{2}$. From this, we see that the diagonal of a square is $\sqrt{2}$ times its side.

Prob. VIII. To find the area of a triangle, having given the base and the altitude.

Formula.—$A=\frac{1}{2}a\times b$.

Rule.—*Multiply the base by the altitude and take half the product.*

I. What is the area of a triangle whose base is 24 feet and altitude 16 feet?

By formula, $A=\frac{1}{2}a\times b=\frac{1}{2}\times16\times24=192$ sq. ft.

II.
1. 24 ft.=base.
2. 16 ft.=altitude.
3. 384 sq. ft.$=16\times24=$product of base and altitude.
4. 192 sq. ft.$=\frac{1}{2}$ of 384 sq. ft.=half the product of the base and the altitude=area.

III. ∴ The area of the triangle is 192 sq. ft.

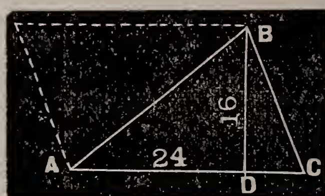

FIG. 7.

Prob. IX. To find the area of a triangle, having given its three sides.

Formula.—$A=\sqrt{s(s-a)(s-b)(s-c)}$, where $s=\frac{1}{2}(a+b+c)$.

***Rule.**—*Add the three sides together and take half the sum; from the half sum, subtract each side seperately; multiply the half sum and the three remainders together and extract the square root of the product.*

**Demonstration.*—In Fig. 7, let $AC=b$, $BC=a$, and $AB=c$. In the right-angled triangle ADB, $BD^2=AB^2-AD^2$, and in the right-angled triangle CDB, $BD^2=BC^2-DC^2$. ∴ $AB^2-AD^2=BC^2-DC^2$, or c^2-

I. What is the area of a triangle whose sides are 13, 14, and 15, feet respectively?

By formula, $A=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{21\times(21-13)\times(21-14)\times(21-15)}=\sqrt{21\times8\times7\times6}=\sqrt{7056}=84$ sq. ft.

II.
1. 42 ft.=13 ft.+14 ft.+15 ft.=sum of the three sides.
2. 21 ft.=$\frac{1}{2}$ of 42 ft.=half the sum of the three sides.
3. 21 ft.—13 ft.=8 ft.=first remainder.
4. 21 ft.—14 ft.=7 ft.=second remainder.
5. 21 ft.—15 ft.=6 ft.=third remainder.
6. $7056=21\times6\times7\times8$=product of half sum and three remainders.
7. 84 sq. ft.= $\sqrt{7056}$=square root of the product of the half sum and three remainders=the area of the triangle.

III. ∴ The area of the triangle is 84 sq. ft.

Prob. X. To find the radius of a circle inscribed in a triangle.

Formula.—$R=2A\div(a+b+c)$.

***Rule.**—*Divide twice the area of the triangle by the sum of the three sides.*

I. Find the radius of a circle inscribed in a triangle whose sides are 3, 4, and 5 feet, respectively.

II.
1. 6 sq. ft.= $\sqrt{s(s-a)(s-b)(s-c)}$=area of the triangle, by formula, Prob. IX.
2. 12 sq. ft.=twice the area of the triangle.
3. 12 ft.=3 ft.+4 ft.+5 ft.=sum of the three sides.
4. ∴ 1 ft.=12÷12=twice the area divided by the sum of the sides=the radius of the inscribed circle.

III. ∴ The radius of the inscribed circle is 1 ft.

$AD^2=a^2-DC^2$, whence $c^2-a^2=AD^2-DC^2$. But $AD^2-DC^2=(AD+DC)(AD-DC)=b(AD-DC)$. ∴ $b(AD-DC)=c^2-a^2$, and $AD-DC=(c^2-a^2)\div b$. But $AD+DC=b$. ∴ By adding the last two equations, we have $2AD=\frac{c^2-a^2}{b}+b=\frac{c^2-a^2+b^2}{b}$; whence $AD=\frac{c^2-a^2+b^2}{2b}$. Since $BD^2=AB^2-AD^2=c^2-AD^2$, if we substitute the value of AD just found, we have $BD^2=c^2-\left(\frac{c^2-a^2+b^2}{2b}\right)^2=\frac{4b^2c^2-(c^2-a^2+b^2)^2}{4b^2}=$
$\frac{(2bc+c^2-a^2+b^2)(2bc-c^2+a^2-b^2)}{4b^2}=\frac{(b^2+2bc+c^2-a^2)[a^2-(b^2-2bc+c^2)]}{4b^2}=$
$\frac{[(b+c)^2-a^2][a^2-(b-c)^2]}{4b^2}$.

∴ $BD=\frac{1}{2b}\sqrt{[(b+c)^2-a^2][a^2-(b-c)^2]},=\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$. Now the area of $ABC=\frac{1}{2}AC\times BD$.

∴ $A=\frac{1}{2}b\times BD=\frac{1}{2}b\times\frac{1}{b2}\sqrt{[(b+c)^2-a^2][a^2-(b-c)^2]}$

$=\frac{1}{4}\sqrt{[(b+c)^2-a^2][a^2-(b-c)^2]}=\frac{1}{4}\sqrt{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}$.

$=\sqrt{s(s-a)(s-b)(s-c)}$, where $2s=(a+b+c)$. *Q. E. D.*

**Note.*—For Demonstration, see any geometry.

Prob. XI. To find the radius of a circle, circumscribed about a triangle whose sides are given.

Formula.—$R=\dfrac{abc}{4A}=\dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$.

***Rule.**—*Divide the product of the three sides by four times the area of the triangle.*

I. What is the radius of a circle circumscribed about a triangle whose sides are 13, 14, and 15 feet, respectively?

II.
1. 2730 cu. ft.$=13\times14\times15=$the product of the three sides.
2. 84 sq. ft.$=\sqrt{s(s-a)(s-b)(s-c)}=$the area of the triangle, by Prob. IX.
3. 336 sq. ft.$=4\times84$ sq. ft.$=$four times the area of the triangle.
4. $8\frac{1}{8}$ ft.$=2730\div336=$the product of the three sides divided by four times the area of the triangle$=$the radius of the circumscribed circle.

III. $\therefore$ The radius of the circumscribed circle is $8\frac{1}{8}$ ft.

Prob. XII. To find the area of an equilateral triangle, having given the side.

Formula.—$A=\frac{1}{4}\sqrt{3}s^2$, where $s=$side. This is what Prob. IX. becomes, when $a=b=c$.

Rule.—*Multiply the square of a side by* $\frac{1}{4}\sqrt{3},=.433013+$.

I. What is the area of an equilateral triangle whose sides are 20 feet?

**Demonstration.*—Let ABC be any triangle, and $ABCE$ the circumscribed circle. Draw the diameter BE, and draw EC. Draw the altitude BD of the triangle ABC. The triangles ADB and BCE are similar, because both are right-angled triangles, and the angle $BAD=$the angle BEC. Hence, $AB:EB::BD:BC$. Hence, $AB\times BC=BE\times BD$ or $ac=2R\times BD$. But, in the demonstration of Prob. IX., we found $BD=\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$.

$\therefore\ ac=2R\times\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$. Whence

$R=\dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}=\dfrac{abc}{4A}$.

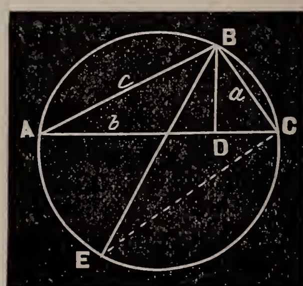

FIG. 8.

By formula, $A=\frac{1}{4}\sqrt{3}\times20^2=100\sqrt{3}=173.205+$sq. ft.

II.
1. 20 ft.=length of a side.
2. 400 sq. ft.$=20^2=$square of a side.
3. 173.205 sq. ft.$=\frac{1}{4}\sqrt{3}\times400=.433013\times400=\frac{1}{4}\sqrt{3}$ times the square of a side,=the area of the triangle.

III. ∴ The area of the equilateral triangle is 173.205+sq. ft.

Prob. XIII. The area and base of a triangle being given, to cut off a triangle containing a given area, by a line running parallel to one of its sides.

Formula.—$b'=b\sqrt{\frac{A'}{A}}$, where A=area of the given triangle; b, the base of the given triangle; and A', the area of the portion to be cut off.

Rule.—*As the area of the given triangle is to the area of the triangle to be cut off, so is the square of the given base to the square of the required base. The square root of the result will be the base of the required triangle.*

I. The area of the triangle ABC is 250 square chains and the base AB, 20 chains; what is the base of the triangle, area equal to 60 sq. chains, cut off by ED parallel to BC?

By formula, $AD=b'=b\sqrt{\frac{A'}{A}}=20\sqrt{\frac{60}{250}}=4\sqrt{6}=9.7979+$ ch.

II.
1. 250 sq. ch.=area of the given triangle ABC.
2. 60 sq. ch.=area of the triangle AED.
3. 20 ch.=base of the triangle ABC.
4. ∴ 250 sq. ch. : 60 sq. ch. :: $20^2 : AD^2$. Whence
5. $AD^2=(400\times60)\div250=96$.
6. ∴ $AD=\sqrt{96}=9.7979+$ch.

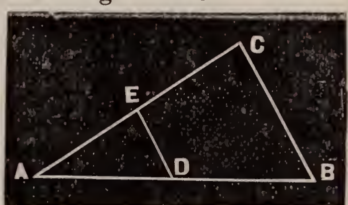

FIG. 9.

III. ∴ The base $AD=9.7979+$ch.

III. TRAPEZOID.

Prob. XIV. To find the area of a trapezoid, having given the parallel sides and the altitude.

Formula.—$A=\frac{1}{2}(b+b')a$, where b and b' are the parallel sides and a, the altitude.

Rule.—*Multiply half the sum of the parallel sides by the altitude.*

I. What is the area of a trapezoid whose parallel sides are 15 meters and 7 meters and altitude 6 meters?

By formula, $A=\frac{1}{2}(b+b')\times a=\frac{1}{2}(15+7)\times 6=66$ m².

II.
1. 7 m.$=DC$, the length of one of the parallel sides, and
2. 15 m.$=AB$, the length of the other side.
3. 22 m.$=7$ m.$+15$ m.$=$sum of the parallel sides.
4. 11 m.$=\frac{1}{2}$ of 22 m.$=$half the sum of the parallel sides.
5. 66 m².$=6\times 11=$area of the trapezoid, $ABCD$.

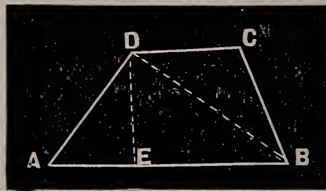

FIG. 10.

III. ∴ The area of the trapezoid is 66 m².

IV. TRAPEZIUM AND IRREGULAR POLYGONS.

Prob. XV. To find the area of a trapezium or any irregular polygon.

Rule.—*Divide the figure into triangles, find the area of the triangles and take their sum.*

I. What is the area of the trapezium $ABCD$, whose diagonal AC is 84 feet, and the perpendiculars DE and BF, 56 and 22 feet, respectively?

II.
1. 84 ft.$=AC=$base of the triangle ADC.
2. 56 ft.$=DE=$ altitude of ADC.
3. ∴ 2352 sq. ft.$=\frac{1}{2}(AC\times DE)=$area of the triangle ADC.
4. 84 ft.$=AC=$base of the triangle ABC.
5. 22 ft.$=BF=$ altitude of ABC.
6. ∴ 924 sq.ft.$=\frac{1}{2}(AC\times BF)$ $=$area of the triangle ABC.
7. 3276 sq. ft.$=$2352 sq. ft.$+$924 sq. ft.$=ADC+ABC=$ area of the trapezium $ABCD$.

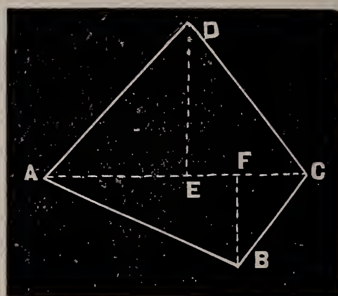

FIG. 11.

III. ∴ The area of the trapezium $ABCD$ is 3276 sq. ft.

V. REGULAR POLYGONS.

Prob. XVI. To find the area of a regular polygon.

Formula.—$A=\frac{1}{2}a\times p$, where p is the perimeter and a, the apothem.

Rule.—*Multiply the perimeter by half the apothem.*

The *Perimeter* of any polygon is the sum of all its sides.

The *Apothem* is the perpendicular drawn from the center to any side of the polygon.

I. What is the area of a regular heptagon whose side is 19.38 and apothem 20?

By formula, $A=\frac{1}{2}a\times p=\frac{1}{2}\times 20\times(7\times 19.38)=1356.6$.

II.
1. 19.38=length of one side.
2. 135.66=length of 7 sides=the perimeter.
3. 20=apothem.
4. $10=\frac{1}{2}$ of 20=half the apothem.
5. $1356.6=10\times 135.66$=product of perimeter by half the apothem.

III. ∴ The area of the heptagon is 1356.6.

Prob. XVII. To find the area of a regular polygon, when the side only is given.

*__Rule.__—*Multiply the square of the side of the polygon by the number standing opposite to its name in the following table of areas of regular polygons whose side is 1:*

Name.	Sides.	Multipliers.
Triangle, - - -	3	$\frac{1}{4}\sqrt{3} = .4330127.$
Tetragon, or square, -	4	$1 = 1.0000000.$
Pentagon, - - -	5	$\frac{5}{4}\sqrt{1+\frac{2}{5}\sqrt{5}} = 1.7204774.$
Hexagon, - - -	6	$\frac{3}{2}\sqrt{3} = 2.5980762.$
Heptagon, - - -	7	$\frac{7}{4}\text{ cot. }\frac{180°}{7} = 3.6339124.$
Octagon, - - - -	8	$2+2\sqrt{2} = 4.8284271.$
Nonagon, - - -	9	$\frac{9}{4}\text{ cot. }20° = 6.1818242.$
Decagon, - -	10	$\frac{5}{2}\sqrt{5+2\sqrt{5}} = 7.6942088.$
Undecagon, - - -	11	$\frac{11}{4}\text{cot. }\frac{180°}{11} = 9.3656399.$
Dodecagon, - - -	12	$3(2+\sqrt{3}) =11.1961524.$

**Demonstration.*—Since a regular polygon can be divided into as many equal isosceles triangles as it has sides, we may find the area of one triangle and multiply this area by the number of triangles, for the whole area. Let ABC be one of these isosceles triangles, taken from a polygon of n sides, AB and BC the equal sides, and AC the base. The angle at the vertex $B=360°\div n$. $A=\frac{1}{2}(180°-360°\div n)=C$. From B let fall a perpendicular on AC at D. Then by trigonometry, $\frac{BD}{\frac{1}{2}AC}=\tan\left(90°-\frac{180°}{n}\right)$. ∴ $BD=\frac{1}{2}AC\cot\left(\frac{180°}{n}\right)$. The area of the triangle $ABC=\frac{1}{2}AC\times BD=\frac{1}{4}AC^2\cot\left(\frac{180°}{n}\right)$. ∴ The area of the polygon$=\frac{n}{4}AC^2\cot\left(\frac{180°}{n}\right)=\frac{n}{4}s^2\cot\left(\frac{180°}{n}\right)$ where s=side. By placing $s=1$, and $n=13$, 14, 15, &c., respectively, the area of polygons of 13, 14, 15, &c., side respectively, may be found.

Prob. XVIII. To find the side of an inscribed square of a triangle, having given the base and the altitude.

Formula.—$s=\dfrac{ab}{a+b}$, where s=side, b the base, and a the altitude.

***Rule.**—*Divide the product of the base and altitude by their sum.*

I. What is the side of an inscribed square of a triangle whose base is 14 feet and altitude 8 feet?

By formula, $s=\dfrac{ab}{a+b}=\dfrac{14\times8}{14+8}=5\frac{1}{11}$ feet.

II.
1. 8 feet=the altitude.
2. 14 feet=the base.
3. 112 sq. ft.$=14\times8$=the product of the base and altitude.
4. 22 feet=14 ft.+8 ft.=their sum.
5. $5\frac{1}{11}$ feet$=112\div22$=the product divided by the sum

III. $\therefore$ $5\frac{1}{11}$ ft.=the side of the inscribed square.

VI. CIRCLE.

Prob. XIX. To find the diameter of a circle, having given the height of an arc and a chord of half the arc.

Formula.—$D=k^2\div a$, in which k=chord of half the arc and a=height.

†**Rule.**—*Divide the square of the chord of half the arc by the height of the chord.*

**Demonstration.*—Let ABC be any triangle whose base is b and altitude a. Produce AC to H, making $CH=BD$. At H, erect the perpendicular HG and make $HG=BD$. Draw AG and at C, erect the perpendicular FC, and draw FK. Then $KE=FC=EN$, and KN is the required inscribed square. For, in the similar triangles AHG and ACF, we have $AH:GH::AC:FC$, or $a+b:a::b:FC$. By inversion, and then by Division, $a:b::a-FC:FC$, or $BI:FC$. In the similar triangles ABC and KBE, $AC:KE::BD:BI$, or $BD:AC::BI:KE$. Whence $a:b::BI:KE$. $\therefore BI:KE::BI:FC$. $\therefore KE=FC$ and the figure KN has its sides equal and its angles right angles by construction. Hence, it is a square. *Q.E.D.*

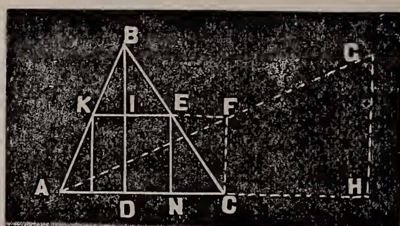

FIG. 12.

†*Demonstration.*— Let $AB=k$, the chord of half the arc ABC, and $BD=a$, the height of the arc ABC. Draw the diameter BE and draw the

I. What is the diameter of a circle of which the height of an arc is 5 m. and the chord of half the arc 10 m.?

By formula, $D=k^2\div a=10^2\div 5=$ 20 m.

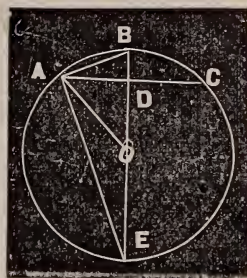

FIG. 13.

II.
1. 10 m.=AB, the length of chord of half the arc.
2. 5 m.=BD, the height of arc.
3. 100 m.^{2}=square of chord.
4. ∴ 20 m.=100÷5=BE, the diameter of the circle.

III. ∴ The diameter of the circle is 20 meters.

Prob. XX. To find the height of an arc, having given the chord of the arc and the radius of the circle.

Formula.—$a=R-\sqrt{R^2-c^2}$, in which, R=radius and $c=\frac{1}{2}$ the chord.

***Rule.**—*From the radius, subtract the square root of the difference of the squares of the radius and half the chord.*

I. The chord of an arc is 12 feet and the radius of the circle is 10 feet. Find the height of the arc.

By formula, $a=R-\sqrt{R^2-c^2}=10-\sqrt{10^2-6^2}=2$ ft.

II.
1. 10 ft.=the radius of the circle.
2. 100 sq. ft.=square of the radius.
3. 12 ft.=the chord.
4. 6 ft.=half the chord.
5. 36 sq. ft.=square of half the chord.
6. 8 ft.$=\sqrt{100-36}=$ square root of the difference of the squares of the radius and half the chord.
7. ∴ 10 ft.—8 ft.=2 ft.=height of the arc.

III. ∴ The height of the chord is 2 feet.

Prob. XXI. To find the chord of half the arc, having given the chord and height of an arc.

Formula.—$k=\sqrt{a^2+c^2}$.

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radius AO. The triangles ADB and BAE are similar, because their angles are equal. Hence, $BE:AB::AB:BD$, or $BE:k::k:a$. Whence $BE=D=k^2\div a$. Q.E D.

N. B.—(1) If a and D are given, $k=\sqrt{D\times a}$; (2) if D and k are given $a=k^2\div D$.

**Demonstration.*—In Fig. 13, we have $BD=BO-DO$. But $DO=\sqrt{AO^2-DA^2}=\sqrt{[R^2-c^2]}$. ∴ $a=R-\sqrt{R^2-c^2}$. If a and R are given, (1) $2c=2\sqrt{[R^2-(R-a)^2]}=2\sqrt{(2aR-a^2)}$; if a and $2c$ are given, (2) $R=(a^2+c^2)\div 2a$.

***Rule.**—*Take the square root of the sum of the squares of the height of arc and half the chord.*

I. Given the chord=48, the height=10, find the chord of half the arc.

By formula, $k=\sqrt{a^2+c^2}=\sqrt{10^2+24^2}=\sqrt{676}=26$.

II.
1. 48=the chord.
2. 576=$\frac{1}{4}$ of 48^2=square of half the chord.
3. 10=height of chord.
4. 100=square of height of chord.
5. 676=576+100=sum of square of half of chord and height.
6. 26=$\sqrt{676}$=square root of sum of square of half of chord and height.

III. ∴ The chord of half the arc is 26.

Prob. XXII. To find the chord of half an arc, having given the chord of the arc and the radius of the circle.

Formula.—$k=\sqrt{2R^2-R\sqrt{4R^2-4c^2}}$.

†**Rule.**—*Multiply the radius by the square root of the difference of the squares of twice the radius and the chord; subtract this product from twice the square of the radius and extract the square root of the difference.*

I. Given the chord of an arc=6 and the radius of the circle =5, find the chord of half the arc.

By formula, $k=\sqrt{2R^2-R\sqrt{4R^2-4c^2}}=\sqrt{2\times5^2-5\sqrt{4\times5^2-6^2}}$ $=\sqrt{10}$.

II.
1. 5=the radius of the circle.
2. 10=twice the radius of the circle.
3. 100=square of twice the radius.
4. 6=chord of the arc.
5. 36=square of the chord.
6. 100—36=64=difference of squares of twice the radius and the chord.
7. 8=$\sqrt{64}$=square root of the above difference.
8. 40=5×8=the product of the above square root and the radius.
9. 50=2×5^2=twice the square of the radius.
10. $\sqrt{50-40}=\sqrt{10}$=chord of half the arc.

III. ∴ The chord of half the arc is $\sqrt{10}$.

**Demonstration.*—In Fig. 13, we have $AB=\sqrt{(AD^2+BD^2)}=\sqrt{[c^2+a^2]}=\sqrt{a^2+c^2}$. $\therefore k=\sqrt{a^2+c^2}$. If k and $2c$ are given, (1)$a=\sqrt{k^2-c^2}$; if k and a are given, (2) $2c=2\sqrt{k^2-a^2}$.

†*Demonstration.*—From Prob. XXI., we have $k=\sqrt{a^2+c^2}$. From Prob. XX. we have $a=R-\sqrt{R^2-c^2}$. $\therefore$ $a^2=2R^2-c^2-2R\sqrt{R^2-c^2}$. Substituting this value of a^2 in the above equation, $k=\sqrt{2R^2-R\sqrt{4R^2-4c^2}}$.

Prob. XXIII. To find the side of a circumscribed polygon, having given the radius of the circle and a side of a similar inscribed polygon.

Formula.—$K'=\dfrac{2KR}{\sqrt{4R^2-K^2}}$, in which K' is the side of the circumscribed polygon and K the side of a similar inscribed polygon.

Rule.—*Divide twice the product of the side of the inscribed polygon and radius by the square root of the difference of the squares of twice the radius and the side of the inscribed polygon.*

I. When $R=1$, find one side of a regular circumscribed dodecagon.

By formula, $K'=\dfrac{2KR}{\sqrt{4R^2-K^2}}=\dfrac{2K}{\sqrt{4-K^2}}$. The formula does not lead to a direct result, since K is not given. But by the formula of Prob. XXI., if k is replaced by K we have $K=\sqrt{2-\sqrt{4-1}}$ for $2c=1$, since it is the side of a regular inscribed hexagon, and $K=\sqrt{2-\sqrt{3}}$, since $2c$ is a side of a regular inscribed dodecagon.

$$\therefore K'=\frac{2K}{\sqrt{4-K^2}}=\frac{2\sqrt{2-\sqrt{3}}}{\sqrt{4-2+\sqrt{3}}}=2\frac{\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}}=.535898.$$

VII. RECTIFICATION OF PLANE CURVES AND QUADRATURES OF PLANE SURFACES.

1. ***To Rectify a Curve*** is to find its length. The term arises from the conception that a right line is to be found which has the same length

2. ***The Quadrature*** of a surface is finding its area. The term arises from the conception that we find a square whose area is equal to the area of the required surface.

The formula for the rectification of plane curves is

$s=\int\sqrt{dx^2+dy^2}=\int\sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}\,dx$, when the curve is referred to rectangular co-ordinates.

$s=\int\sqrt{\left\{1+r^2\left(\dfrac{d\theta}{dr}\right)^2\right\}}\,dr$, or $s=\int\sqrt{\left\{r^2+\left(\dfrac{dr}{d\theta}\right)^2\right\}}\,d\theta$, when the curve is referred to polar co-ordinates.

$$\left.\begin{aligned}
s&=\int\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2\right\}}\,dx\\
s&=\int\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}}\,dy\\
s&=\int\sqrt{\left\{1+\left(\frac{dx}{dz}\right)^2+\left(\frac{dy}{dz}\right)^2\right\}}\,dz
\end{aligned}\right\}$$

are formulæ for the rectification of curves of double curvature, when referred to rectangular co-ordinates.

$$\left.\begin{array}{l} s=\int\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2+r^2\sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2\right\}}\,d\theta \\ s=\int\sqrt{\left\{1+r^2\left(\frac{d\theta}{dr}\right)^2+r^2\sin^2\theta\left(\frac{d\varphi}{dr}\right)^2\right\}}\,dr \\ s=\int\sqrt{\left\{r^2\sin^2\theta+\left(\frac{dr}{d\varphi}\right)^2+r^2\left(\frac{d\theta}{d\varphi}\right)^2\right\}}\,d\varphi \end{array}\right\}$$

are formulæ for the rectification of curves of double curvature, referred to polar co-ordinates.

$A=\int ydx$ or $\int xdy$ is the formula for the quadrature of any plane surface referred to rectangular co-ordinates.

$A=\int\frac{1}{2}r^2d\theta$ is the formula for the quadrature of plane surfaces, referred to polar co-ordinates.

3. ***A Surface of Revolution*** is the surface generated by a line (right or curved) revolving around a fixed right line as an axis, so that sections of the volume generated, made by a plane perpendicular to the axis are circles.

$S=2\pi\int y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx$ is the formula for a surface of revolution, referred to rectangular co-ordinates.

$S=2\pi\int yds=2\pi\int r\sin\theta\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta$ is the formula for a surface of revolution, referred to polar co-ordinates.

$V=\pi\int y^2dx$ or x^2dy is the formula for the volume of a solid of revolution referred to rectangular co-ordinates.

$V=\int\int\int dxdydz$ and $V=\int\int zdxdy$ are formulæ for the cubature of solids, requiring triple and double integration.

$V=\int\int zrd\theta dr$ and $V=\int\int\int r^2\sin\theta d\varphi d\theta dr$ are the formulæ for cubature of solids referred to polar co-ordinates. From the equation to the surface of the solid, z must be expressed as a function of r and θ.

$x^2+y^2=R^2$ is the rectangular equation of a circle referred to the center.

$y^2=2Rx-x^2$ is the rectangular equation of a circle referred to the left hand vertex as origin of co-ordinates.

$r=2R\cos.\theta$ is the equation of the circle referred to polar co-ordinates.

Prob. XXIV. To find the circumference of a circle, the radius being given.

Formula.—$C.=4\int_0^R\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx=4\int_0^R\left\{\frac{x^2+y^2}{y^2}\right\}^{\frac{1}{2}}dx=4\int_0^R\frac{-Rdx}{(R^2-x^2)^{\frac{1}{2}}}=4R(1+\frac{1}{2.3}+\frac{1.3}{2.4.5}+\frac{1.3.5}{2.4.6.7}+\frac{1.3.5.7}{2.4.6.8.9}+\&c)=4R\times1.570796+=3.141592\times2R=2\pi R$, in which $\pi=3.141592+$. Since the diameter is twice the radius, we have $2\pi R=\pi D$, in which D is the diameter. $\therefore$ $C.=2\pi R=\pi D$. $\therefore$ (1) $R=\frac{C}{2\pi}$, (2) $D=\frac{C}{\pi}$, where C is the circumference.

Rule.—*Multiply twice the radius or the diameter by 3.141592.*

I. What is the circumference of a circle whose radius is 17 rods?

By formula, $C=2\pi R=3.141592\times34$ rods $=106.814128$ rods.

II. 1. 17 rods=the radius.
2. 34 rods=2×17 rods=the diameter.
3. 106.814128 rods=3.141592×34 rods=the circumference.

III. $\therefore$ The circumference is 106.814128 rods.

Note.—The ratio of the circumference to the diameter can not be exactly ascertained. An untold amount of mental energy has been expended upon this problem; but all attempts to find an exact ratio have ended in utter failure. Many persons not noted along any other line, claimed to have found this *clavem impossibilitibus* by which they have unlocked all the difficulties that have encumbered the quadrature of the circle for more than two thousand years. The Quadrature of the Circle is to find a square whose area shall be exactly equal to that of the circle. This can not be done, since the ratio of the circumference to the diameter can not be exactly ascertained. Persons claiming to have held communion with the "gods" and extorted from them the exact ratio are ranked by mathematicians in the same class with the inventors of Perpetual Motion and the discoverers of the Elixir of Life, Alkahest, the Fountain of Perpetual Youth, and the Philosopher's Stone. Lambert, an Alsacian mathematician, proved, in 1761, that this ratio is incommensurable. In 1881, Lindemann, a German mathematician, demonstrated that this ratio is transcendental, and that the quadrature of the circle by means of the ruler and compasses only, or by means of any algebraic curve, is *impossible*. Its value has been computed to several hundred decimal places. Archimedes, in 287 B. C., found it to be between $3\frac{10}{71}$ and $3\frac{1}{7}$; Metius, in 1640, gave a nearer approximation in the fraction $\frac{355}{113}$; and, in 1873, Mr. W. Shank presented to the Royal Society of London a computation extending the decimal to 707 places. The following is its value to 600 decimal places:

3. 141, 592, 653, 589, 793,238,462,643,383, 279, 502,884, 197, 169, 399,375,105,820,974,944,592,307,816,406,286,208,998,628,034,825, 342,117,067,982,148,086,513,282,306,647,093,844,609,550,582,231, 725,359,408,128,481,117,450,284,102,701,938,521,105,559,644,622, 948,954,930,381,964,428,810,975,665,933,446,128,475,648,233,786,

783,165,271,201,909,145,648,566,923,460,348,610,454,326,648,213. 393,607,260,249,141,273,724,587,006,606,315,588,174,881.520,920, 962,829,254,091,715,364,367,892,590,360,011,330,530,548,820,466, 521,384,146,951,941,511,609,433,057,270,365,759,591,953,092,186, 117,381,932,611,793,105,118,548,074,462,379.834,749,567,351,885, 752,724,891,227,938,183.011.949.129.833.673.362.441.936.643.086, 021,395.016.092.448,077.230.943.628.553.096,620,275,569,397,986, 950,222,474,996,206,074.970.304.123,669+.

$\frac{1}{4}\pi=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+$&c.

$\frac{1}{6}\pi^2=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+$&c.* Bernoulli's Formula.

$\frac{1}{2}\pi=\left[\frac{2.4.6.8.10....\&c.}{3.5.7.9.11....\&c.}\right]^2$ .... Wallis's Formula, 1655.

$$\frac{1}{2}\pi=1+\cfrac{1}{1+\cfrac{1.2}{1+\cfrac{2.3}{1+\cfrac{3.4}{1+\cfrac{1}{1+\cfrac{4.5}{1+}}}}}}\ \ldots\ldots$$ Sylvester's Formula, 1869.

$$\frac{4}{\pi}=1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ldots\ldots}}}}$$ Buckner's Formula.

The Greek letter π, was first used by Euler, to designate the ratio of the circumference to the diameter.

Prob. XXV. To find the length of any arc of a circle, having given the chord of the arc and the height of the arc, i. e., the versed sine of half the arc.

(a). ***Formula.***—$s=\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int\left(\frac{x^2+y^2}{y^2}\right)dx=$ $-\int\frac{Rdx}{(R^2-x^2)^{\frac{1}{2}}}=R\sin^{-1}\frac{x}{R}=R\left[\frac{x}{R}+\frac{x^3}{2.3.R^3}+\frac{1.3x^5}{2.4.5.R^5}+\right.$ $\left.\frac{1.3.5x^7}{2.4.6.7.R^7.}+\&c.\right],=\frac{1}{2a}(a^2+c^2)\left[\frac{c^2-a^2}{a^2+c^2}+\frac{1}{6}\frac{(c^2-a^2)^3}{(a^2+c^2)^3}+\right.$ $\left.\frac{3}{40}\frac{(c^2-a^2)^5}{(a^2+c^2)^5}+\frac{5}{112}\left(\frac{c^2-a^2}{a^2+c^2}\right)^7+\&c.\right]$, in which a=the altitude of the arc and c=half the chord of the arc.

**Note.*—This series was discovered by Bernoulli, but he acknowledged his inability to sum it. Euler found the result to be $\frac{1}{6}\pi^2$. For an interesting discussion of the various formulæ for π, see *Squaring the Circle, Britannica Encyclopedia*

(*b*.) ***Formula.***—$s=\text{arc}=2\sqrt{(a^2+c^2)}\times\left[1+\dfrac{10a^2}{60c^2+33a^2}\right]$.

This formula is a very close approximation to the true length of the arc when a and c are small. The first formula may be extended to any desired degree of accuracy.

Rule from (*b*).—*Divide 10 times the square of the height of the arc by 15 times the square of the chord and 33 times the height of the chord; multiply this quotient increased by 1, by 2 times the square root of the sum of the squares of the height and half the chord.*

I. The chord of an arc is 25, and versed-sine 15, required the length of the arc.

By formula (*a*), $arc=\dfrac{a^2+c^2}{2a}\left[\dfrac{c^2-a^2}{a^2+c^2}+\frac{1}{6}\left(\dfrac{c^2-a^2}{a^2+c^2}\right)^3+\dfrac{3}{40}\left(\dfrac{c^2-a^2}{a^2+c^2}\right)^5+\dfrac{5}{112}\left(\dfrac{c^2-a^2}{a^2+c^2}\right)^7+\&c.\right]=\dfrac{15^2+25^2}{2\times15}\left[\dfrac{25^2-15^2}{15^2+25^2}+\dfrac{1}{6}\times\left(\dfrac{25^2-15^2}{15^2+25^2}\right)^3+\dfrac{3}{40}\left(\dfrac{25^2-15^2}{15^2+25^2}\right)^5+\dfrac{5}{112}\left(\dfrac{25^2-15^2}{15^2+25^2}\right)^7+\&c.\right]=53.58+$ ft.

II.
1. 25 ft.=length of the chord.
2. 15 ft.=height of the arc, or the versed-sine.
3. 2250 sq. ft.=10 times 15^2=10 times the square of the height of the arc.
4. 9375 sq. ft.=15 times 25^2=15 times the square of the chord.
5. 7425 sq. ft.=33 times 15^2=33 times the square of the height of arc.
6. 17800 sq. ft.=7425 sq. ft.+9375 sq. ft.
7. $\frac{45}{356}=2250\div17800=10$ times $15^2\div(15$ times 25^2+33 times $15^2)$.
8. $1+\frac{45}{356}=\frac{401}{356}=1+10$ times $15^2\div(15$ times 25^2+33 times $15^2)$.
9. $381\frac{1}{4}$ sq. ft.$=15^2+(12\frac{1}{2})^2$.
10. 53.58 ft.$=\frac{401}{356}\times\sqrt{15^2+(12\frac{1}{2})^2}=\frac{401}{356}\times\frac{5}{2}\sqrt{61}$=length of arc, nearly.

III. ∴ 53.58 ft.=length of the arc.

Prob. XXVI. To find the area of a circle having given the radius, diameter, or circumference.

Formula.—$A=4\int y\,dx,=4\int_0^R(R^2-x^2)^{\frac{1}{2}}dx=4\left[\frac{1}{2}\times(R^2-x^2)+\frac{1}{2}R^2\sin^{-1}\frac{x}{R}\right]_0^R=2R^2\left[\frac{x}{R}+\frac{x^3}{2.3R^3}+\frac{1.3.x^5}{2.4.5.R^5}+\frac{1.3.5.\ x^7}{2.4.6.7R^7}+\&c=\frac{1}{2}\pi\right]=\pi R^2=\frac{1}{4}\pi D^2=\frac{C^2}{4\pi}=\frac{1}{2}R\times C$, when the ra-

dius and circumference are given. $\therefore$ (1) $R=\sqrt{A\div\pi}$, (2) $D=\sqrt{4A\div\pi}=2R=2\sqrt{A\div\pi}$, and (3) $C=\sqrt{4\pi A}=2\sqrt{\pi A}$.

Rule I.—*The area of a circle equals the square of the radius multiplied by 3.141592; or (2) the square of the diameter multiplied by .785398; or (3) the square of the circumference multiplied by .07958; or (4) the circumference mnltiplied by $\frac{1}{4}$ of the diameter; or (5) the circumference multiplied by $\frac{1}{2}$ of the radius.*

Rule II.—*Having given the area. (1) To find the radius: Divide the area by 3.141592, and extract the square root of the quotient. (2) To find the diameter: Divide the area by 3.141592 and multiply the square root of the quotient by 2. (3) To find the circumference: Multiply the area by 3.141592 and multiply the square root of the product by 2.*

I. What is the area of a circle whose radius is 7 feet?

By formula, $A=\pi R^2=3.141592\times7^2=153.93804+$ sq. ft.

II. $\begin{cases}1.\ 7 \text{ ft.}=\text{the radius.}\\ 2.\ 49 \text{ sq. ft.}=7^2=\text{square of the radius.}\\ 3.\ 153.93804 \text{ sq. ft.}=3.141592\times49 \text{ sq. ft.}=\text{area of the circle.}\end{cases}$

III. $\therefore$ 153.93804 sq. ft.=area of the circle.

I. What is the area of a circle whose diameter is 4 rods?

By formula, $A=\frac{1}{4}\pi D^2=\frac{1}{4}\times3.141592\times4^2=12.566368$ sq. ft.

II. $\begin{cases}1.\ 4 \text{ ft.}=\text{the diameter.}\\ 2.\ 16 \text{ sq. ft.}=\text{square of the diameter.}\\ 3.\ 12.566368 \text{ sq. ft.}=\frac{1}{4}\times3.141592\times4^2=.785398\times16 \text{ sq. ft.}\\ \quad=\text{area of the circle.}\end{cases}$

III. $\therefore$ 12.566368 sq. ft.=area of the circle.

I. What is the area of a circle whose circumference is 5 meters?

By formula, $A=\frac{C^2}{4\pi}=\frac{25^2}{4\pi}=1.989 \text{ m}^2$.

II. $\begin{cases}1.\ 5 \text{ m.}=\text{the circumference.}\\ 2.\ 25 \text{ m.}^2=\text{the square of the circumference.}\\ 3.\ 1.989 \text{ m.}^2=.07958\times25 \text{ m.}^2=\text{the area of the circle.}\end{cases}$

III. $\therefore$ 1.989 m.^{2}=the area of the circle.

Remark.—We might have found the radius by formula (1) under Prob. XXIV and then applied the first of Rule I. above. We might have found the radius by formula (1) of Prob. XXIV and then applied (5) of Rule I. above.

I. What is the circumference of a circle whose area is 10 A.?

By formula (3), $C=2\sqrt{\pi A}=2\sqrt{3.141592\times1600}=80\sqrt{\pi}80\times$ $1.7724539=141.796312$ rods.

II. 1. 10 A.$=1600$ sq. rds.$=$the area of the circle.
2. $1600\div\pi=$the square of the radius.
3. $\therefore 40\sqrt{\frac{1}{\pi}}=\frac{40}{\pi}\sqrt{\pi}=$the radius.
4. $\frac{80}{\pi}\sqrt{\pi}=\pi 2$ times $\frac{40}{\pi}\sqrt{\pi}=$the diameter.
5. $80\sqrt{\pi}=\pi\times\frac{40}{\pi}\sqrt{\pi}=141.796312$ rods$=$the circumference.

III. $\therefore$ 141.796312 rods$=$the circumference of the circle.

I. With what length of rope must a horse be tied to a stake so that he can graze over one acre of grass and no more?

By formula (1), $R=\sqrt{A\div\pi}=\sqrt{160\div\pi}=4\sqrt{\frac{10}{\pi}}=7.1364+$ rd.

II. 1. 1 A.$=160$ sq. rd.$=$area of the circle over which the horse can graze.
2. $160\div\pi=$square of the radius.
3. $\sqrt{160\div\pi}=4\sqrt{10\div\pi}=7.1364$ rd.$=$radius or length of rope.

III. $\therefore$ 7.1364 rd.$=$length of the rope.

Prob. XXVII. To find the area of a sector, or that part of a circle which is bounded by any two radii and their included arc, having given the chord of the arc and the height of the arc.

$$*\textit{Formula.}—A=\int ydx=2\int(R^2-x^2)^{\frac{1}{2}}dx=x(R^2-x^2)^{\frac{1}{2}}+R^2\sin^{-1}\frac{x}{R}=\frac{c^2-a^2}{2a}\left\{\left[\frac{a^2+c^2}{2a}\right]^2-\left[\frac{c^2-a^2}{2a}\right]^2\right\}^{\frac{1}{2}}+\left[\frac{a^2+c^2}{2a}\right]^2\sin^{-1}\frac{c^2-a^2}{c^2+a^2}=c\left[\frac{c^2-a^2}{2a}\right]+\left[\frac{a^2+c^2}{2a}\right]^2\sin^{-1}\frac{c^2-a^2}{c^2+a^2}=c\left[\frac{c^2-a^2}{2a}\right]+\left[\frac{a^2+c^2}{2a}\right]^2\left\{\frac{c^2-a^2}{c^2+a^2}+\frac{1}{1.2.3}\left[\frac{c^2-a^2}{c^2+a^2}\right]^3+\frac{1}{1.2.3.4.5}\left[\frac{c^2-a^2}{c^2+a^2}\right]^5+\frac{1}{1.2.3.4.5.6.7}\left[\frac{c^2-a^2}{c^2+a^2}\right]^7+\&c.\right\}$$

in which c is half the chord of the arc and a the height of arc.

Demonstration.—Let $AB=x$, $BD=y$, and $R=AD=$the radius of the circle. Then $x^2+y^2=R^2$, the equation of the circle referred to the center. Now $A=2\int ydx$; but $y=(R^2-x^2)^{\frac{1}{2}}$, from the equation of the circle.

$\therefore A=2\int(R^2-x^2)^{\frac{1}{2}}dx=x(R^2-x^2)^{\frac{1}{2}}+R^2\ sin^{-1}\frac{x}{R}$. But $x=R-a$ and $y=c$.

Hence $A=(R-a)[R^2-(R-a)^2]+R^2\sin^{-1}\frac{R-a}{R}$. But, from (2) Prob. XX, R

Rule.—(1) *Find the length of the arc by Problem XXV, and then multiply the arc by half the radius which may be found by Problem XX, in which c and a are known and R is the unknown quantity.*

(2) *If the arc is given in degrees, take such a part of the whole area of the circle as the number of degrees in the arc is of 360°.*

I. Find the area of the sector, the chord of whose arc is 40 feet, and the versed-sine of half the arc 15 feet.

By formula, $A=c\left(\frac{c^2-a^2}{2a}\right)+\left(\frac{a^2+c^2}{2a}\right)^2\left\{\frac{c^2-a^2}{c^2+a^2}+\frac{1}{1.2.3}\left(\frac{c^2-a^2}{c^2+a^2}\right)^3+\&c.\right\}=20\left(\frac{20^2-15^2}{30}\right)+\left(\frac{15^2+20^2}{30}\right)^2\left\{\frac{20^2-15^2}{20^2+15^2}+\frac{1}{1.2.3}\left(\frac{20^2-15^2}{20^2+15^2}\right)^3+\frac{1}{1.2.3.4.5}\left(\frac{20^2-15^2}{20^2+15^2}\right)^5+\&c.\right.=558.125$ sq. ft.

II.
1. 53.58 ft.$=2(15^2+20^2)^{\frac{1}{2}}\times\left\{1+\frac{10\times15^2}{60\times20^2+33\times15^2}\right.=$ length of the arc, by (*b*), Prob. XXV.
2. $20\frac{5}{6}$ ft.$=\frac{a^2+c^2}{2a}$=radius of the circle, by solving the formula of Prob. XX with respect to R.
3. $\therefore$ 558.125 sq. ft.$=\frac{1}{2}(20\frac{5}{6}\times53.58)$=area of the sector.

III. $\therefore$ 558.125 sq. ft.=the area of the sector.

$=\frac{a^2+c^2}{2a}$. Hence, $A=\left(\frac{c^2-a^2}{2a}\right)c+\left(\frac{c^2-a^2}{2a}\right)^2\sin^{-1}\left(\frac{c^2-a^2}{c^2+a^2}\right)$ But, from Trigonometry, we have $\sin^{-1}\theta=\theta-\frac{1}{1.2.3}\theta^3+\frac{1}{1.2.3.4.5}\theta^5-\&c.$ Hence, $A=\left(\frac{c^2-a^2}{2a}\right)c+\left(\frac{c^2-a^2}{2a}\right)^2\left\{\frac{c^2-a^2}{c^2+a^2}-\frac{1}{1.2.3}\left(\frac{c^2-a^2}{c^2+a^2}\right)^3+\frac{1}{1.2.3.4.5}\left(\frac{c^2-a^2}{c^2+a^2}\right)^5-\&c.\right\}$

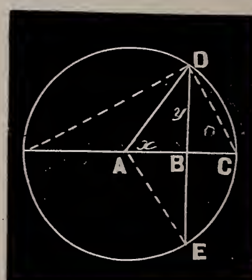

FIG. 14.

In this formula, $\left(\frac{c^2-a^2}{2a}\right)c$ is the area of the triangle DEA. For $x=R-a=\frac{a^2+c^2}{2a}-a=\frac{c^2-a^2}{2a}=$ the altitude and c is half the base of the triangle. $\therefore\left(\frac{c^2-a^2}{2a}\right)c=$ the area of the triangle DEA. Therefore, if we subtract the area of the triangle DEA from the area of the sector, we shall have the area of the segment DEC. Hence, the area of the segment DEC is $\left(\frac{c^2-a^2}{2a}\right)^2\left\{\left(\frac{c^2-a^2}{c^2+a^2}\right)-\frac{1}{1.2.3}\left(\frac{c^2-a^2}{c^2+a^2}\right)^3+\frac{1}{1.2.3.4.5}\left(\frac{c^2-a^2}{c^2+a^2}\right)^5-\&c.\right\}$. This result may be carried to any desired degree of accuracy.

I. What is the area of a sector whose arc is 40° and the radius of the circle 9 feet?

II. $\begin{cases} 1.\ 9\text{ ft.}=\text{radius of the circle.} \\ 2.\ \pi R^2=\pi 9^2=\text{area of the whole circle.} \\ 3.\ 40^\circ=\text{length of the arc.} \\ 4.\ 40^\circ=\frac{1}{9}\text{ of }360^\circ. \\ 5.\ \pi 9=\frac{1}{9}\text{ of }\pi 9^2=28.274328\text{ sq. ft.}=\text{area of the sector.} \end{cases}$

III. ∴ The area of the sector is 28.274328 sq. ft.

Prob. XXVIII. To find the area of the segment of a circle, having given the chord of the arc and the height of the segment, i. e., the versed-sine of half the arc.

Formula.—(*a*) $A=\left(\frac{c^2-a^2}{2a}\right)^2\left\{\left(\frac{c^2-a^2}{c^2+a^2}\right)-\frac{1}{1.2.3}\left(\frac{c^2-a^2}{c^2+a^2}\right)^3+\frac{1}{1.2.3.4.5}\left(\frac{c^2-a^2}{c^2+a^2}\right)^5-\&c.\right\}$

(*b*) $A=\frac{a^3}{2(2c)}+\frac{2}{3}(2ca)=\frac{a^3}{4c}+\frac{4ca}{3}$.

***Rule.**—*Divide the cube of the height by twice the base and increase the quotient by two-thirds of the product of the height and base.*

I. What is the area of a segment whose base is 2 feet and altitude 1 foot?

By formula (*b*), $A=\frac{a^3}{2(2c)}+\frac{2}{3}(2ca)=\frac{1^3}{2\times2}+\frac{2}{3}(2\times1)=1\frac{7}{12}$sq.ft.

**Demonstration.*—In the last figure, let $BC=a=$altitude of the segment and $DE=2c=$the base of the segment. Then $BD^2=BC\times BF=a(2R-a)=c^2$. Whence $R=\frac{c^2+a^2}{2a}$, and $AD=R-a=\frac{c^2+a^2}{2a}-a=\frac{c^2-a^2}{2a}$. $DC=k=\sqrt{c^2+a^2}$. By Trigonometry, $\frac{BD}{AD}=\sin\angle DAC$, or $\frac{c}{R}=\sin\angle DAC$. Now $2\pi R=360^\circ$. $\therefore R=\frac{180^\circ}{\pi}$. Therefore, $R:arcDC::\frac{180^\circ}{\pi}:?=\frac{arcDC}{R}\times\frac{180^\circ}{\pi}$. Let $s=arc$ DCE. Then the $\angle DAC=\frac{\frac{1}{2}s}{R}=\frac{s}{2R}$. $\therefore\frac{c}{R}=\sin\frac{s}{2R}$. In like manner, from the right angled triangle FDC, $\frac{DC}{FC}=\sin\angle CFD$, or since the $\angle CFD=$the $\frac{1}{2}\angle CAD$, $\frac{k}{2R}=\sin\frac{s}{4R}$. Now since the sine of any angle $\theta=\theta-\frac{1}{1.2.3}\theta^3+\frac{1}{1.2.3.5}\theta^5-\frac{1}{1.2.3.5.7}\theta^7+\&c.$, the above equation becomes

$\frac{c}{R}=\frac{s}{2R}-\frac{1}{1.2.3}\left(\frac{s}{2R}\right)^3+\frac{1}{1.2.3.5}\left(\frac{s}{2R}\right)^5-\&c.$.....(1), and

$\frac{k}{2R}=\frac{s}{4R}-\frac{1}{1.2.3}\left(\frac{s}{4R}\right)^3+\frac{1}{1.2.3.5}\left(\frac{s}{4R}\right)^5-\&c.$.....(2). Multiplying equation (2) by 8 and subtract equation (1) in order to eliminate the term containing s^3, we have approximately, $\frac{4k-c}{s}=\frac{3s}{2R}-\frac{3}{4}\left(\frac{1}{1.2.3.5}\right)\left(\frac{s}{2R}\right)^5+\&c.$ Omitting the negative quantity, since it is very small in comparison with s and because it is still more diminished by a succeeding positive quantity, we have

II.
1. 1 ft.=altitude of the segment.
2. 2 ft.=base of the segment.
3. 4 ft.=twice the base of the segment.
4. 1 cu. ft.=cube of the height of the segment.
5. $\frac{1}{4}$ sq. ft.=1÷4=quotient of the cube of the height and twice the base.
6. 2 sq. ft.=2×1=product of the height and base.
7. $1\frac{1}{3}$ sq. ft.=$\frac{2}{3}$ of the product of the height and base.
8. ∴ $1\frac{1}{3}$ sq. ft.+$\frac{1}{4}$ sq. ft.=$1\frac{7}{12}$sq. ft.=area of the segment.

III. ∴ The area of the segment is $1\frac{7}{12}$ sq. ft.

Prob. XXIX. To find the area of a circular zone, or the space included between any two parallel chords and their intercepted arcs.

Formula.—(*a*) $$A=\left(\frac{c^2-a^2}{2a}\right)^2\left\{\frac{c^2-a^2}{c^2+a^2}-\frac{1}{1.2.3}\left(\frac{c^2-a^2}{c^2+a^2}\right)^2+\frac{1}{1.2.3.4.5}\left(\frac{c^2-a^2}{c^2+a^2}\right)^5-\&c.\right\}-\left(\frac{c'^2-a'^2}{2a'}\right)^2\left\{\frac{c'^2-a'^2}{c'^2+a'^2}-\frac{1}{1.2.3}\left(\frac{c'^2-a'^2}{c'^2+a'^2}\right)^3+\frac{1}{1.2.3.4.5}\left(\frac{c'^2-a'^2}{c'^2+a'^2}\right)^5-\&c.\right\}.$$

(*b*) $$A=\frac{a^3}{2(2c)}+\tfrac{2}{3}(2ca)-\left[\frac{a'^3}{2(2c')}+\tfrac{2}{3}(2c'a')\right]$$

Rule.—*Find the area of each segment by Prob. XXVIII., and take the difference between them, if both chords are on the same side of the center; if on opposite sides of the center, subtract the sum of the areas of the segments from the whole area of the circle.*

I. What is the area of a zone, one side of which is 96, and the other 60, and the distance between them 25?

Let $AB=60=2c'$, $CD=96=2c$, and $HK=25=h$. Then $AH=30=c'$ and $CK=48=c$. Let $OA=R$. Then $OH=\sqrt{R^2-c'^2}$, $OK=\sqrt{R^2-c^2}$. But $OH=OK+HK$; ∴ $\sqrt{R^2-c'^2}=\sqrt{R^2-c^2}+h$. Whence $R=\frac{1}{2h}\sqrt{4c^2h^2+(c^2-h^2-c'^2)^2}$.

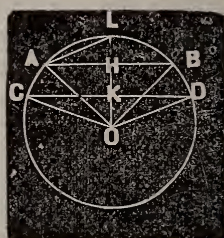

FIG. 15.

$\therefore LH=a'=R-OH=R-\sqrt{(R^2-c^2)}=\frac{1}{2h}\sqrt{4c^2-h^2+(c^2-h^2-c'^2)^2}-\frac{1}{2h}\sqrt{4h^2(c^2-c'^2)}$

$S=\frac{8k-2c}{3}=\frac{8\sqrt{c^2+a^2}-2a}{3}=\tfrac{2}{3}(4\sqrt{c^2+a^2}-a)$. This is the approximate length of an arc in terms of its height and base. Now the area of the segment $DCE=\frac{1}{2}AC\times arc\ DCE$—area of the triangle $DEA=\frac{1}{2}R\times S-\frac{1}{2}AB\times DE=\frac{1}{2}\left(\frac{c^2+a^2}{2a}\right)^2\times\tfrac{2}{3}(4\sqrt{c^2+a^2}-a)-\frac{1}{2}\left(\frac{c^2-a^2}{2a}\right)c=\frac{1}{6a}[4(c^2+a^2)^{\frac{3}{2}}-4c^3+2ca^2]=\frac{1}{6a}[\sqrt{(16c^6+48c^4a^2+48c^2a^4+16a^6)}-4c^3+2ca^2]=\frac{1}{6a}\left[\left(4a^3+6c^2a+\frac{3a^4}{2c}\right)\right.$ nearly$\left.-4c^3+2ca^2\right]=\frac{1}{6a}\left(8ca+\frac{3a^4}{2c}\right)=\frac{a^3}{2(2c)}+\tfrac{2}{3}(2ca)$. *Q. E. D.*

$\overline{+(c^2-h^2-c'^2)^2}$. In like manner, $LK=a=R-\sqrt{R^2-c^2},=\frac{1}{2h}$
$\sqrt{4c^2h^2+(c^2-h^2-c'^2)^2}-\frac{1}{2h}(c^2h^2-c'^2).^2$

$\therefore$ By formula (b), $A=\frac{a^3}{2(2c)}+\frac{2}{3}(2ca)-\left[\frac{a'^3}{2(2c')}+\frac{2}{3}(2c'a')\right],=$
$\left\{\frac{1}{2h}\left[\sqrt{4c^2h^2+(c^2-h^2-c'^2)^2}-(c^2-h^2c'^2)^2\right]\right\}^3\div 2.2c+\frac{2}{3}\times 2c\times\frac{1}{2h}$
$\left[\sqrt{4c^2h^2+(c^2-h^2-c'^2)^2}-(c^2-h^2-c'^2)^2\right]-\left[\left\{\frac{1}{2h}\sqrt{4c^2h^2+(c^2\right.\right.$
$\left.\overline{-h^2-c'^2)^2}-\sqrt{4h^2(c^2-c'^2)+(c^2-h^2-c'^2)^2}\right]\Big\}^3\div 2(2c)+\frac{2}{3}\times 2c'$
$\left.\times\frac{1}{2h}\left[\sqrt{4h^2c^2+(c^2-h^2-c'^2)^2}-\sqrt{4h^2(c^2-c'^2)+(c^2-h^2-c'^2)^2}\right]\right\}$
$=2547-408\frac{1}{3}=2138\frac{2}{3}$.

II. 1. $50=R=\frac{1}{2h}\sqrt{4c^2h^2+(c^2-h^2-c'^2}=\frac{1}{50}\sqrt{4\times 48^2\times 25^2+}$ $\overline{(48^2-25^2-30^2)}$=radius of the circle.
2. $OK=\sqrt{R^2-c^2}=\sqrt{50^2-48^2}=14$.
3. $\therefore$ $LK=a=50-14=36$=altitude of segment CLD.
4. $OH=\sqrt{R^2-c'^2}=\sqrt{50^2-30^2}=40$.
5. $\therefore$ $LH=a'=50-40=10$=altitude of the segment ALB.
6. $\therefore$ $\frac{36^3}{2\times 96}+\frac{2}{3}(96\times 36)=2547$=area of segment $CDBLA$.
7. $\frac{10^3}{2\times 60}+\frac{2}{3}(60\times 10)=408\frac{1}{3}$=area of the segment ABL.
8. $\therefore$ $2547-408\frac{1}{3}=2138\frac{2}{3}$=area of the zone $CDBA$.

III. $\therefore$ $2138\frac{2}{3}$=area of the zone $ABDC$.

Note.—This result is only approximately correct. The radius of the circle may be found by the following rule:

Subtract half the difference between the two half chords from the greater half-chord, multiply the remainder by said difference, divide the product by the width of the zone, and add the quotient to half the width. To the square of this sum add the square of the less half chord, and take the square root of the sum.

This rule is derived from the formula in the above solution, in which

$$R=\sqrt{\frac{h^2c^2+(c^2-h^2-c'^2)^2}{4h^2}}=\sqrt{\left[\frac{(c^2+h^2-c'^2)^2+4h^2c'^2}{4h^2}\right]},$$

Prob. XXX. To find the area of a circular ring, or the space included between the circumference of two concentric circles.

Formula.—(*a.*) $A=\pi\ (R^2-r^2)$, in which R and r are the radii of the circles.

(*b.*) $^*A=\frac{1}{4}\pi c^2$, in which c is a chord of the larger circle tangent to the smaller circle.

I. Required the area of a ring the radii of whose bounding circles are 9 and 7 respectively.

By formula (a), $A=\pi(R^2-r^2)=\pi(9^2-7^2)=32\pi=$ 100.530944.

II.
1. $9=R=$radius of the larger circle, and
2. $7=r=$radius of the smaller circle.
3. $\pi 9^2=\pi R^2=$area of larger circle, and
4. $\pi 7^2=\pi r^2=$area of smaller circle.
5. $\therefore\ \pi 9^2-\pi 7^2=\pi(9^2-7^2)=32\pi=$ 100 530944=area of the ring.

III. $\therefore$ 100.530944=the area of the ring.

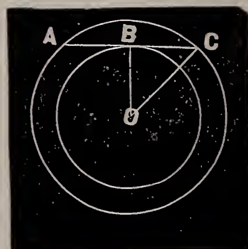

FIG. 16.

**Demonstration.*—Let ABC be the chord of the large circle, which is tangent to the smaller circle, and let $ABC=c$. Then $BC=\frac{1}{2}c=\sqrt{(OC^2-OB^2)}=\sqrt{(R^2-r^2)}$. $\therefore\ \frac{1}{4}c^2=R^2-,r^2$ and $\frac{1}{4}\pi c^2=\pi(R^2-r^2)$ But $\pi(R^2-r^2)$ is the difference of the areas of the two circles or the area of the ring $\therefore\ \frac{1}{4}\pi c^2=$the area of the ring. *Q. E. D.*

Prob. XXXI. To find the areas of circular lunes, or the spaces between the intersecting arcs of two eccentric circles.

$$\textit{Formula.}\text{—}A=\frac{a'^3}{2(2c)}+\tfrac{2}{3}(2ca)-\left[\frac{a'^3}{2(2c)}+\tfrac{2}{3}(2ca')\right].$$

Rule.—*Find the area of the two segments of which the lunes are formed, and their difference will be the area required.*

I. The chord AB is 20, and the height DC is 10, and DE 2; find the area of the lune $AEBC$.

$$=\sqrt{\left[\left(\frac{c^2-c'^2}{2h}+\tfrac{1}{2}h^2\right)+c'^2\right]}=\sqrt{\left\{\left[\frac{\left[\frac{c-c'}{2}\right](c-c')}{h}+\tfrac{1}{2}h\right]^2+c'^2.\right\}}.$$

If now we find the altitudes of the two segments and then find the length of the arcs of the segments by formula (b), Prob. XXV, and then find the area of the sectors by multiplying the length of the arcs by half the radius, from the areas of the sectors subtract the triangles formed by the radii of the circles and the chord of the arcs, we shall thenhave the area of the two segments. Taking their difference, we shall have for the area of the zone 2136.75, which is a nearer approximation to the true area.

By formula, $A=\frac{a}{2(2c)}+\frac{2}{3}(2ca)-\left[\frac{a'^3}{2(2c)}+\frac{2}{3}(2ca')\right]=\frac{10^3}{2\times20}$
$+\frac{2}{3}(20\times10)-\left[\frac{2^3}{2\times20}+\frac{2}{3}(20\times2)\right]=131\frac{7}{15}$.

II.
1. AB=chord=20.
2. DE = height of segment $AEBD$=2. [$ACBD$=10
3. DC = height of segment
4. $\frac{10^3}{2\times20}+\frac{2}{3}(20\times10)=158\frac{1}{3}$= area of the segment $ACBD$.
5. $\frac{2^3}{2\times20}+\frac{2}{3}(20\times2)=26\frac{13}{15}$= area of the segment $AEBD$.

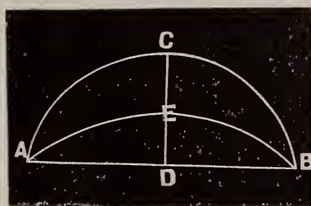

FIG. 17.

III. $\therefore$ $158\frac{1}{3}-26\frac{13}{15}=131\frac{7}{15}$=area of the lune $ACBE$.

VIII. CONIC SECTIONS.

DEFINITIONS.

1. ***The Conic Sections*** are such plane figures as are formed by the cutting of a cone.

2. If a cone be cut through the vertex, by a plane which also cuts the base, the sections will be a *triangle*.

3. If a right cone be cut in two parts, by a plane parallel to the base, the section will be a *circle*.

4. If a cone be cut by a plane which passes through its two slant sides in an oblique direction, the section will be an *ellipse*.

5. ***The Transverse Axis*** of an ellipse is its longest diameter.

6. ***The Conjugate Axis*** of an ellipse is its shortest diameter.

7. ***An Ordinate*** is a right line drawn from any point of the curve, perpendicular to either of the diameters.

8. ***An Abscissa*** is that part of the diameter which is contained between the vertex and the *ordinate*.

9. ***A Parabola*** is a section formed by passing a plane through a cone parallel to either of its slant sides.

10. ***The Axis*** of a parabola is a right line drawn from the vertex, so as to divide the figure into two equal parts.

11. ***The Ordinate*** is a right line drawn from any point in the curve perpendicular to the axis.

12. ***The Abscissa*** is that part of the axis which is contained between the vertex and the ordinate.

13. ***An Hyperbola*** is a section formed by passing a plane through a cone in a direction to make an angle at the base greater than that made by the slant height. It will thus pass through the symmetrical opposite cone.

14. ***The Transverse Diameter*** of an hyperbola, is that part of the axis intercepted between the two opposite cones.

15. ***The Conjugate Diameter*** is a line drawn through the center perpendicular to the transverse diameter

16. ***An Ordinate*** is a line drawn from any point in the curve perpendicular to the axis.

17. ***The Abscissa*** is the part of the axis intercepted between that ordinate and the vertex.

1. ELLIPSE.

$a^2y^2+b^2x^2=a^2b^2$ is the equation of an ellipse referred to the center.

$y^2=\frac{b^2}{a^2}(2ax-x^2)$ is the equation of the ellipse referred to left hand vertex.

In these equations, a is the semi-transverse diameter and b the semi-conjugate diameter; y is any ordinate and x is the corresponding abscissa. When any three of these quantities are given the fourth may be found by solving either of the above equations with reference to the required quantity.

$\rho=\frac{a(1-e^2)}{1-e\cos\theta}$ is the polar equation referred to the centre, and

$\rho=\frac{a(1-e^2)}{1+e\cos\theta}$ is the polar equation referred to the left hand vertex.

Prob. XXXII. To find the circumference of an ellipse, the transverse and conjugate diameters being known.

Formula.—cir.$=C=4\int\sqrt{dy^2+dx^2}=4\int\sqrt{\frac{x^2}{y^2}(1-c^2)^2 dx^2+dx^2}=4\int\frac{\sqrt{y^2+x^2(1-e^2)^2}}{y}dx=4\int\sqrt{\frac{a^2-e^2x^2}{a^2-x^2}}dx=$

$4\int_0^a\frac{x}{\sqrt{a^2-x^2}}\sqrt{1-\frac{e^2x^2}{a^2}}dx=4\left(a\sin^{-1}\frac{x}{a}-\frac{e^2}{2a}\left\{\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}\right\}-\frac{e^4}{2.4a^3}\left[\frac{3a^2}{2}\left\{\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}-\frac{x^3}{4}\right.\right.\right.$

$$\sqrt{a^2-x^2}\Big]-\frac{3e^6}{2.4.6a^5}\Big\{\frac{5a^2}{6}\Big[\frac{3A^2}{2}\Big(\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}\Big)-$$

$$\frac{x^3}{4}\sqrt{a^2-x^2}\Big]-\frac{x^5}{6}\sqrt{a^2-x^2}\Big\}-\&c.\Big)_o^a=4\Big(\frac{\pi a}{2}-\frac{e^2}{2a}\Big(\frac{a}{2}\cdot\frac{\pi a}{2}\Big)-$$

$$\frac{e^4}{2.4a^3}\Big[\frac{3a^2}{4}\Big(\frac{a}{2}\cdot\frac{\pi a}{2}\Big)\Big]-\frac{3e^6}{2.4.6a^5}\Big\{\frac{5a^2}{6}\Big[\frac{3a^2}{4}\Big(\frac{a}{2}\cdot\frac{\pi a}{2}\Big]\Big\}\Big)-\&c.\Big)=$$

$$2\pi a\Big\{1-\frac{e^2}{2.2}-\frac{3e^4}{2.2.4.4}-\frac{3.3.5.e^6}{2.2.4.4.6.6}-\&c,\Big\}\text{ in which }e=\frac{\sqrt{a^2-b^2}}{a^2}.$$

Rule.—*Multiply the square root of half the sum of the squares of the two diameters by 3.141592, and the product will be the circumference, nearly.*

I. What is the circumference of an ellipse whose axes are 24 and 18 feet respectively?

$$\text{By formula, } Cir.=C=2\pi\times12\Big\{1-\frac{1}{2.2}\Big(1-\frac{9^2}{12^2}\Big)^2-\frac{3}{2.2.4.4}\Big(1-\frac{9^2}{12^2}\Big)^4-\&c.=2\pi\times12\times.87947.=66.31056\text{ ft., nearly.}$$

II.
1. 576 sq. ft.$=24^2=$square of the transverse diameter.
2. 324 sq. ft.$=18^2=$square of the conjugate diameter.
3. 900 sq. ft.=sum of the squares of the diameters.
4. 450 sq. ft.=half the sum of the squares of the diameters
5. $15\sqrt{2}$ft.$=\sqrt{450}=$square root of half the sum of the squares of the diameters.
6. $\pi15\sqrt{2}$ ft.= 66.6434 ft., nearly, =the circumference of the ellipse.

III. ∴ The circumference of the ellipse is 66.6434 ft. nearly, by the rule.

Prob. XXXIII. To find the length of any arc of an ellipse, having given the ordinate, abscissa, and either of the diameters.

Formula.—$s=2\big[\tfrac{1}{2}\pi a\big\{1-(\tfrac{1}{2})^2\frac{e^2}{1}-(\tfrac{1}{2}\cdot\tfrac{3}{4})^2\frac{e^4}{3}-(\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6})^2\frac{e^6}{5}-\&c\big\}\big]-a\sin^{-1}\frac{x}{a}-\frac{e^2}{2a}\Big\{\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}\Big\}-\frac{e^4}{2.4a^3}\Big[\frac{3a^2}{4}\Big\{\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}-\frac{x^3}{4}\sqrt{a^2-x^2}\Big]\Big\}-\&c.$, in which x is the abscissa; a the semi-transverse diameter; and $e=\sqrt{\frac{a^2-b^2}{a^2}}=$the eccentricity of the ellipse.

Rule.—*Find the length of the quadrant CB by Prob. XXXI and CF by substituting the value of x in the above series. Twice the difference between these arcs will give the length of the arc FBG.*

I. What is the length of the arc FBG, if $OE=x=9$, $EF=y=8$, and $OC=b=10$?

Since $a^2y^2+b^2x^2=a^2b^2$, we find, by substituting the values of x, y, and b, $a=15$. Then by the formula,

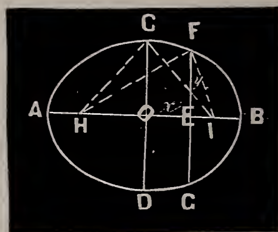

FIG. 18.

$$FBG=s=2\left[\tfrac{1}{2}\pi a\left\{1-(\tfrac{1}{2})^2\frac{e^2}{1}-\left\{\frac{1}{2}\cdot\frac{3}{4}\right\}^2\frac{e^4}{3}-\left\{\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right\}^2\frac{e^6}{5}-\&c.\right\}\right.$$
$$-a\sin^{-1}\frac{x}{a}-\frac{e^2}{2a}\left\{\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}\right\}$$
$$-\frac{e^4}{2.4a^3}\left[\frac{3a^2}{4}\left\{\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}-\right.\right.$$
$$\left.\left.\frac{x^3}{4}\sqrt{a^2-x^2}\right\}\right]-\left.\&c.\right]=\pi15\left\{1-(\tfrac{1}{2})^2\frac{e^2}{1}\right.$$
$$\left.-(\tfrac{1}{2}\cdot\tfrac{3}{4})^2\frac{e^4}{3}-(\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6})^2\frac{e^6}{5}-\&c.\right\}-2\left\{15\sin^{-1}\frac{9}{15}\right.$$
$$-\frac{e^2}{2.15}\left\{\frac{15^2}{2}\sin^{-1}\frac{9}{15}-\frac{9}{2}\sqrt{15^2-9^2}\right\}-\frac{e^4}{2.4.15^3}\left[\frac{3.15^2}{4}\left\{\frac{15^2}{2}\sin^{-1}\frac{9}{15}-\right.\right.$$
$$\left.\left.\frac{9}{2}\sqrt{15^2-9^2}-\frac{9^3}{4}\sqrt{15^2-9^2}\right]-\left.\&c.\right\}=\pi15\times.815-2\left\{\frac{37}{12}\pi-\frac{e^2}{30}\right.$$
$$\left.\left(\frac{185}{8}\pi-72\right)-\frac{e^4}{8.15^3}\left[\frac{3.15^2}{4}\left\{\frac{185}{8}\pi-72-3^7\right]-\&c.\right\}=38.406-.$$

Prob. XXXIV. To find the area of an ellipse, the transverse and conjugate diameters being given.

Formula.—$A=4\int ydx=4\,\dfrac{b}{a}\displaystyle\int_0^a(\sqrt{a^2-x^2})dx=\pi ab$, in which a and b are the semi-transverse, and semi-conjugate diameters.

Rule.—*Multiply the product of the semi-diameters by $\pi=3.141592$, or multiply the product of the diameters by $\frac{1}{4}\pi=.785398$.*

I. What is the area of an ellipse whose traverse diameter is 70 feet and conjugate diameter 50 feet?

By formula, $A=\pi ab=\pi35\times25=2748.893$ sq. ft.

II. $\begin{cases}\text{1. 35 ft.}=\frac{1}{2}\text{ of 70 ft.}=\text{length of the semi-transverse diameter.}\\ \text{2. 25 ft.}=\frac{1}{2}\text{ of 50 ft.}=\text{length of the semi-conjugate diameter.}\\ \text{3. }\therefore\text{ 2748.893 sq. ft.}=\pi\times35\times25=\text{the area of the ellipse.}\end{cases}$

III. $\therefore$ The area of the ellipse is 2748.893 sq. ft.

NOTE.—$\pi ab=\sqrt{\pi a^2.\pi b^2}$. $\therefore$ The area of an ellipse is a mean proportional between the circumscribed and inscribed circles.

Prob. XXXV. To find the area of an elliptic segment, having given the base of the segment, its height, and either diameter of the ellipse, the base being parallel to either diameter.

Formulae.—(*a*) $A=\int ydx$, or $\int xdy=\frac{b}{a}\int x(a^2-x^2)^{\frac{1}{2}}dx,=$

$ab[1-\frac{1}{2}\left(\frac{x}{a}\right)^2-2\left(\frac{x}{2a}\right)^4-4\left(\frac{x}{2a}\right)^6-2.5\left(\frac{x}{2a}\right)^8-2^2.7\left(\frac{x}{2a}\right)^{10}-2^2.3.7$
$\left(\frac{x}{2a}\right)^{12}-\&c.]$, or $\frac{a}{b}\int(b^2-y^2)^{\frac{1}{2}}dy=ab[1-\frac{1}{2}\left(\frac{y}{b}\right)^2-2\left(\frac{y}{2b}\right)^4-4$
$\left(\frac{y}{2b}\right)^6-2.5\left(\frac{y}{2b}\right)^8-2^2.7\left(\frac{y}{2b}\right)^{10}-2^2.3.7\left(\frac{y}{2b}\right)^{12}-\&c.]$

(b) $A=\int ydx=\frac{b}{a}\left[x(a^2-x^2)^{\frac{1}{2}}+a^2\sin^{-1}\frac{x}{a}\right]$.

The former formula of (*a*) gives the area of a segment whose base is parallel to the conjugate diameter and the latter the area of a segment whose base is parallel to the transverse diameter.

Rule.—*Find the area of the corresponding segment of the circle described upon the same axis to which the base of the segment is perpendicular. Then this axis is to the other axis as the area of the circular segment is to the area of the elleptic segment.*

2. PARABOLA.

$y^2=2px$ is the rectangular equation of the parabola.

$\rho=\frac{p}{1-\cos\theta}$ is the polar equation of the parabola.

In the rectangular equation, $HG=y$, the ordinate; $AG=x$, the abscissa; $AF=AE=\frac{1}{2}p$. If any two of these are given the remaining one may be found from the equation. p is a constant quantity.

Prob. XXXVI. To find the length of any arc of a parabola cut off by a double ordinate.

Formula.—$s=2\int\sqrt{dy^2+dx^2}=\frac{2}{p}\int(p^2+y^2)^{\frac{1}{2}}dy=$

$\frac{y}{p}\sqrt{p^2+y^2}+p\log[y+\sqrt{p^2+y^2}]+C=\frac{y}{p}\sqrt{p^2+y^2}+$

$p\log\left[\frac{y+\sqrt{p^2+y^2}}{p}\right]$, or $\frac{2}{p}\int(p^2+y^2)^{\frac{1}{2}}dy=2\,(y+\frac{1}{2}\cdot\frac{1}{3}\frac{y^3}{p^2}-$

$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{5}\frac{y^5}{p^4}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{7}\frac{y^7}{p^6}-\&c)$.

Rule.—*When the abscissa is less than half the ordinate: To the square of the ordinate add $\frac{4}{3}$ of the square of the abscissa and twice the square root of the sum will be the length of the arc.*

I. What is the length of the arc KAH, if AG is 2 and GH 6?

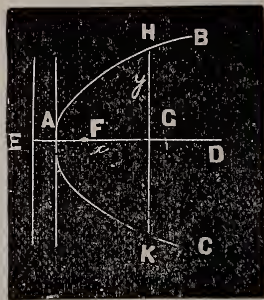

FIG. 19.

By formula, $s=\frac{y}{p}\sqrt{p^2+y^2}+p\log\left[\frac{y+\sqrt{p^2+y^2}}{p}\right]=\frac{6}{p}\sqrt{p^2+6^2}+p\log\left[\frac{6+\sqrt{p^2+6^2}}{p}\right]$. Since $y^2=2px$, we have $p=\frac{y^2}{2x}=\frac{36}{4}=9$. $\therefore s=\frac{2}{3}(3\sqrt{13})+9\log\left[\frac{6+3\sqrt{13}}{9}\right]=2\sqrt{13}+9\log\frac{1}{3}(2+\sqrt{13})$, or

by series, $s=2(y+\frac{1}{2}\cdot\frac{1}{3}\frac{y^3}{p^2}-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{5}\frac{y^5}{p^4}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{7}\frac{y^7}{p^6}$—&c.]$=12.7105$ $=$length of the arc, nearly.

II.
1. $2=AG=$the abscissa.
2. $6=GH=$the ordinate.
3. $36=$the square of the ordinate.
4. $\frac{16}{3}=\frac{4}{3}$ of $2^2=\frac{4}{3}$ of the square of the abscissa.
5. $2\sqrt{(\frac{16}{3}+36)}=12.858=$the length of the arc, nearly.

III. $\therefore$ $12.858=$length of the arc, nearly.

Prob. XXXVII. To find the area of a parabola, the base and height being given.

Formula.—$A=2\int ydx=2\int\frac{1}{p}y^2dy=\frac{2y^3}{3p}=\frac{4}{3}xy=\frac{2}{3}(x.\ 2y)$, i. e., the area of parabola HKA is $\frac{2}{3}$ of the circumscribed rectangle.

Rule.—*Multiply the base by the height and $\frac{2}{3}$ of the product will be the area.*

I. What is the area of a parabola whose double ordinate is 24m. and altitude 16m.?

By formula, $A=\frac{2}{3}(x.\ 2y)=\frac{2}{3}(16\times24)=256\text{m}^2$.

II.
1. 24m.$=HK$(in last figure)=the double ordinate, or base of the parabola.
2. 16m.$=AG=$the altitude of the parabola.
3. $\therefore 384\text{m}^2=16\times24=$the area of the rectangle circumscribed about the parabola.
4. $\frac{2}{3}$ of $384\text{m}^2=256\text{m}^2=$the area of the parabola.

III. $\therefore$ The area of the parabola is 256m^2.

Prob. XXXVIII. To find the area of a parabolic frustum, having given the double ordinates of its ends and the distance between them.

Formula.—$A=\frac{2}{3}a\times\frac{B^3-b^3}{B^2-b^2}$, in which a is the distance between the double ordinates, B the greater and b the lesser double ordinate.

Rule.—*Divide the difference of the cubes of the two ends by the difference of their squares and multiply the quotient by $\frac{2}{3}$ of the altitude.*

I. What is the area of a parabolic frustum whose greater base is 10 feet, lesser base 6 feet, and the altitude 4 feet?

By formula, $A=\frac{2}{3}a\times\frac{B^3-b^3}{B^2-b^2}=\frac{2}{3}\times4\times\frac{10^3-6^3}{10^2-6^2}=32\frac{2}{3}$ sq. ft.

II.
1. 10 ft.=the greater base,
2. 6 ft.=the lesser base, and
3. 4 ft.=the altitude.
4. 784 cu. ft.$=10^3-6^3$=the difference of the cubes of the two bases.
5. 64 sq. ft.$=10^2-6^2$=the difference of the squares of the two bases.
6. $12\frac{1}{4}$ ft.$=784\div64$=the quotient of the difference of the cubes by the difference of the squares.
7. $\therefore\ \frac{2}{3}\times(4\times12\frac{1}{4})=32\frac{2}{3}$ sq. ft.=the area of the frustum.

III. $\therefore$ The area of the frustum is $32\frac{2}{3}$ sq. ft.

3. HYPERBOLA.

1. $a^2y^2-b^2x^2=-a^2b^2$ is the equation of thc hyperbola referred to its axes in terms of its semi-axes.

2. $y^2=-\frac{b^2}{a^2}(2ax-x^2)$ is the equation of a hyperbola referred to its transverse axis and a tangent at the left hand vertex.

3. $\rho=\frac{a(1-e^2)}{1-e\cos\theta}$ is the polar equation of the hyperbola.

Having given any three of the four quantites a, b, x, y, the other may be found by solving the rectangular equation with reference to the required quantity.

Prob. XXXIX. To find the length of any arc of an hyperbola, beginning at the vertex.

Formula.—$s=\sqrt{dy^2+dx^2}=\sqrt{\left(\frac{(a^2+b^2)y^2+b^4}{b^2(y^2+b^2)}\right)}dx=$
$y\left(1+\frac{1}{1.2.3}\frac{a^2x^2}{b^4}-\frac{1.1.3}{1.2.3.4.5}\frac{a^4+4a^2b^2}{b^8}y^4+\frac{1.1.3.3.5}{1.2.3.4.5.6.7}\right.$
$\left.\frac{a^6+4a^4b^2+8a^2b^4}{b^{12}}y^6-\&c.\right)$

Rule.—*1. Find the parameter by dividing the square of the conjugate diameter by the transverse diameter.*

2. To 19 times the transverse, add 21 times the parameter of the axis, and multiply the sum by the quotient of the abscissa divided by the transverse.

3. To 9 times the transverse, add 21 times the parameter, and multiply the sum by the quotient of the abscissa divided by the transverse.

4. To each of the products thus found, add 15 times the parameter, and divide the former by the latter; then this quotient being multiplied by the ordinate will give the length, nearly.

(*Bonnycastle's Rule.*)

NOTE.—*A parameter* is a double ordinate passing through the focus.

I. In the the hyperbola DAC, the transverse diameter $GA=80$, the conjugate $HI=60$, the ordinate $DC=10$, and the abscissa $AB=2.1637$; what is the length of the arc DAC?

By formula, $DAC=$

$$s=2x\left(1+\frac{1}{1.2.3}\frac{a^2x^2}{b^4}-\frac{1.1.3}{1.2.3.4.5}\frac{a^4+4a^2b^2}{b^8}x^4+\frac{1.1\,3.3.5}{1.2.3.4.5.6.7}\frac{a^6+4a^4b^2+8x^2b^4}{b^{12}}\right)=$$

20.658.

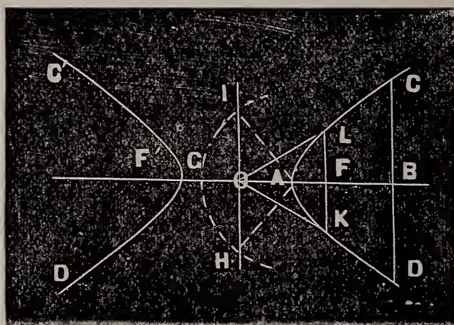

FIG. 20.

II.
1. $45=2\overline{OI}^2\div OA=2b^2\div a=$the parameter LK which, in the figure, should be drawn to the right of DC, to be consistent with the nature of the problem.
2. $1520=19\times80=19$ times the transverse diameter.
3. $945=21\times45=21$ times the parameter.
4. $2465=$sum of these two products.
5. $.02704=2.1637\div80=$quotient of abscissa and transverse diameter.
6. $2465\times.02704=66.6536=$sum of the *products* multiplied by the said quotient. Also,
7. $[(80\times9)+(45\times21)]\times\frac{2.1637}{80}=(720+945)\times.02704=45.0216$. Whence
8. $(15\times45+66.6536)\div(15\times45+45.0216)=741.6536\div720.0216=1.03004$.
9. $\therefore 1.03004\times10=10.3004=$length of the arc AC, nearly.

III. $\therefore$ The length of the arc is 10.3004.

Prob. XL. To find the area of an hyperbola, the transverse and conjugate axes and abscissa being given.

Formulae.—$(a) A=2\int y dx=2\frac{b}{a}\int_a^{x'}(x^2-a^2)^{\frac{1}{2}}dx=\frac{b}{a}x'\sqrt{x'^2-a^2}-ab\log\left[\frac{x'+\sqrt{x'^2-a^2}}{a}\right]=x'y'-ab\log\left[\frac{x'+\sqrt{x'^2-a^2}}{a}\right]$; or, (b) $A=4xy\left\{\frac{1}{3}-\frac{1}{1.3.5}\frac{x^2}{x^2+x^2}-\frac{1}{3.5.7.}\left(\frac{x^2}{a^2+x^2}\right)^2-\frac{1}{5.7.9}\left(\frac{x^2}{a^2+x^2}\right)^3-\&c.\right\}$

Rule.—*1. To the product of the transverse diameter and abscissa, add $\frac{5}{7}$ of the square of the abscissa, and multiply the square root of the sum by 21.*

2. Add 4 times the square root of the product of the transverse diameter and abscissa, to the product last found and divide the sum by 75.

3. Divide 4 times the product of the conjugate diameter and abscissa by the transverse diameter, and this last quotent multiplied by the former will give the area required, nearly.—Bonnycastle's Rule.

I. If, in the hyperbola DAC, the transverse axis AG is 30, the conjugate diameter HI, 18 and the abscissa AB, 10; what is the area of the hyperbola DAC?

By formula (a), $A=x'y'-ab\log_e\left[\frac{x'+\sqrt{x'^2-a^2}}{a}\right]=25y'-15\times9\log_e\left[\frac{25+\sqrt{26^2-15^2}}{15}\right]=300-135\log_e\left[\frac{25+\sqrt{400}}{15}\right]=300-135\log_e 3=300-135\times1.09861228=151.687343$, y' being found from the equation $a^2y'^2-b^2x'^2=-a^2b^2$, in which $a=15$, $b=9$ and $x'=15+10=25$.

II.
1. *1.* $21\sqrt{30\times10+\frac{5}{7}\times10^2}=21\sqrt{300+500\div7}=21\sqrt{371.42857}=21\times19.272=404.712$, by the first part of the rule.
2. *2.* $(4\sqrt{30\times10}+404.712)\div75=(4\times17.3205+404.712)\div75=6.3199$, by the second part of the rule.
3. *3.* $\therefore\frac{18\times10\times4}{30}\times6.3199=151.6776$, by the third part of the rule, $=$the area of the hyperbola, nearly.

III. $\therefore$ 151.6776=the area of the hyperbola.

Prob. XLI. To find the area of a zone of an hyperbola.

$$\textit{Formula.}\text{—}A=2\frac{b}{a}\int_{x_1}^{x_2}(x^2-a^2)^{\frac{1}{2}}dx$$

$$=\frac{b}{a}x_2\sqrt{x_2^2-a^2}-ab\log_e\left[\frac{x_2+\sqrt{x_2^2-a^2}}{a}\right]-\frac{b}{a}x_1\sqrt{x_1^2-a^2}+$$

$$ab\log_e\left[\frac{x_1+\sqrt{x_1^2-a^2}}{a}\right]=x_2y_2-x_1y_1-ab\log_e\left[\frac{x_2+\sqrt{x_2^2-a^2}}{a}\right]+$$

$$ab\log_e\left[\frac{x_1+\sqrt{x_1^2-a^2}}{a^2}\right]=x_2y_2-x_1y_1-ab\log_e\left[\frac{x_2+\sqrt{x_2^2-a^2}}{x_1+\sqrt{x_1^2-a^2}}\right],$$

in which (x_2, y_2), and (x_1, y_1) are the co-ordinates of the points C and L respectively.

I. What is the area of a zone of an hyperbola whose transverse diameter is $2a=10$ feet, conjugate diameter $2b=6$ feet, the lesser double ordinate of the zone being 8 feet and the greater 12 feet?

By formula, $A=x_2y_2-x_1y_1-ab\log_e\left\{\frac{x_2+\sqrt{(x_2^2-{}^2)}}{x_1+\sqrt{(x_1^2-a^2)}}\right\}$

$$=6x_2-4x_1-15\log_e\left(\frac{bx_2+ay_2}{bx_1+ay_1}\right)=6x_2-4x_1-15\log_e\left(\frac{3x_2+30}{3x_1+20}\right),$$

But from the equation, when $y=y_2=6$, $x=x_2=10\sqrt{6}$ and when $y=y_1=4$, $x=x_1=13\frac{1}{3}$. Substituting these values of x_2 and y_2, we have $A=50\sqrt{6}-66\frac{2}{3}-30\log_e\left(\frac{30\cdot\sqrt{6}+30}{50+20}\right)$

$$=\left\{50\sqrt{6}-66\tfrac{2}{3}-30\log_e[\tfrac{3}{7}(\sqrt{6}+1)]\right\}\text{ sq. ft.}$$

Prob. XLII. To find the area of a sector of an hyperbola, KALO.

$$\textit{Formula.}\text{—}A=ab\log_e\left(\frac{x}{a}+\frac{y}{b}\right).$$

Rule.—*Find the area of the segment AKL by Prob. XL., and subtract it from the area of the triangle KOL.*

I. What is the area of the sector OAL (Fig. 20) if $OA=a=5$, $OI=b=3$, and $LF=y=4$?

By formula, $A=\frac{1}{2}ab\log_e\left(\frac{x}{a}+\frac{y}{b}\right)=7\frac{1}{2}\log_e(\frac{1}{5}x+\frac{4}{3})=7\frac{1}{2}$ $\log_e[\frac{1}{15}(3x+20)]$. But when $y=4$, $x=13\frac{1}{3}$. Hence, $A=7\frac{1}{2}\log_e(\frac{14}{3})$.

IX. HIGHER PLANE CURVES.

1. ***Higher Plane Curves*** are loci whose equations are above the second degree, or which involve transcendental functions, *i. e.*, a function whose degree is infinite.

I. THE CISSOID OF DIOCLES.

1. ***The Cissoid of Diocles*** is the curve generated by the vertex of a parabola rolling on an equal parabola.

2. If pairs of equal ordinates be drawn to the diameter of a circle, and through one extremity of this diameter and the point in the circumference through which one of the ordinates is let fall, a line be drawn, the locus of the intersection of this line and the equal ordinate, or that ordinate produced is the *Cissoid of Diocles.*

3. $y^2=\frac{x^3}{2a-x}$ is the equation of the cissoid referred to rectangular axes.

$\rho=2a\sin\theta\tan\theta$ is the polar equation of the curve.

Prob. XLIII. To find the length of an arc OAP of the cissoid.

Formula.—$s=OAP=$

$$\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=$$

$$\int\sqrt{1+\left(\frac{(3a-x)\sqrt{x}}{\sqrt{(2a-x)^3}}\right)^2}dx=$$

$$a\int\sqrt{\frac{8a-3x}{(2a-x)^3}}dx=a\left\{\sqrt{\frac{8a-3x}{2a-x}}\right.$$

$$-2+3\log_e$$

$$\left.\left[\frac{\sqrt{2a}(\sqrt{3}+2)}{\sqrt{3}\sqrt{2a-x}+\sqrt{8a-3x}}\right]\right\}$$

I. What is the length of the arc *OAN*, in which case $x=a$?

By formula, $s=a\left\{\sqrt{5}-\right.$

$$\left.2+3\log_e\left[\frac{\sqrt{2}(\sqrt{3}+2)}{\sqrt{3}+\sqrt{5}}\right]\right\}$$

Fig. 21.

Prob. XLIV. To find the area included between the curve and its asymptote, BM.

Formula.—$A=2\int_{0}^{2a} y\,dx=2\int_{0}^{2a}\sqrt{\frac{x^3}{2a-x}}\,dx=\left[-\frac{1}{2}\sqrt{x}.\sqrt{2a-x}\,(a+x)-3\sin^{-1}\frac{\sqrt{2a-x}}{\sqrt{2a}}\right]_{0}^{2a}=3\pi a^2$, *i. e.*, 3 times the area of the circle, OEB.

Note.—The name Cissoid is from the Greek κισσοειδης, like ivy, from κισσος, ivy, ειδος, form. The curve was invented by the Greek geometer Diocles, A D. 500, for the purpose of solving two celebrated problems of the higher geometry; viz., to trisect a plane angle, and to construct two geometrical means between two given straight lines. The construction of two geometrical means between two given straight lines is effected by the *cissoid.* Thus in the figure of the cissoid, ED and OG are the two geometrical means between the straight lines OD and PG: that is, $OD:ED::OG:PG$. The trisection of a plane angle is effected by the conchoid. The duplication of the cube, *i. e.*, to find the edge of a cube whose volume shall be twice that of a cube whose volume is given, may be effected by the cissoid. Thus, on KC lay off $CH=2BC$, and draw BH. Let fall from the point F, where BH cuts the curve, the perpendicular FR. Then $RF=2BR$. Now a cube described on RF is twice one described on OR; for, since $FR=y, OR=x$, and $BR=2a-x$, we have $RF^2=\frac{OR^3}{BR}=\frac{OR^3}{\frac{1}{2}FR}$, or $\frac{1}{2}RF^3=OR^3$. $\therefore$ $FR^3=2OR^3$ Q. E. D.

2. THE CONCHOID OF NICOMEDES.

1. ***The Conchoid*** is the locus formed by measuring, on a line which revolves about a fixed point without a given fixed line, a constant length in either direction from the point where it intersects the given fixed line.

2. $x^2y^2=(b+y)^2(a^2-y^2)$ is the equation of the *conchoid* referred to rectangular axes.

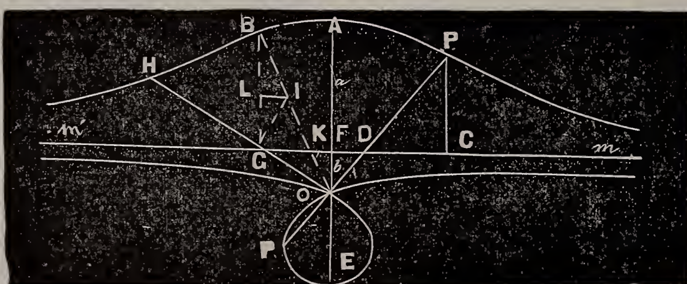

FIG. 22.

3. $\rho=b\sec\theta\pm a$ is the polar equation referred to polar co-ordinates. In this equation, θ is the angle PO makes with AO.

Prob. XLV. To find the length of an arc of the conchoid.

Formula.—$s=\int\sqrt{1+\left(\frac{dr}{d\theta}\right)^2}\,\theta d=\int\sqrt{1+\tan^2\theta\sec^2}\,d\theta.$

Prob. XLVI. To find the whole area of the cinchoid between two radiants each making an angle θ with OA.

Formula.—$A=2\int\frac{1}{2}r^2d\theta=b^2\int(\sec\theta\pm a)^2d\theta=$ $b^2\tan\theta+2a^2\theta+3b\sqrt{a^2-b^2}$ or $b^2\tan\theta+2a^2\theta$, according as a is or is not greater than b. 1. The area above the directrix $m\,m'$ and the same radiants $=2ab\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+a^2\theta$.

2. The area of the loop which exists when a is $>b$ is $a^2\cos^{-1}\frac{b}{a}-2ab\log\left\{\frac{b+\sqrt{a^2-b^2}}{b-\sqrt{a^2-b^2}}\right\}+b\sqrt{a^2-b^2}$.

NOTE.—The name conchoid is from the Greek, κογχοειδηϛ, from κόγχη, shell, and ειδοϛ, form, and signifies shell-form. It was invented by the Greek geometer Nicomedes, about A. D. 100 for the purpose of trisecting any plane angle. The trisection of an angle may be accomplished by this curve as follows: Let AOH be any angle to be trisected. From any point, G, in one side let fall a perpendicular, GF, upon the other. Take $AF=2GO$, and with O as the fixed point, $m\,m'$ as the fixed line and PP as the revolving line of which $PD=a$ is constant, construct the arc of the conchoid, PAH. Erect BG perpendicular to $m\,m'$ and draw BO. Then is BOA one third of HOA. For bisect BK at I, and draw GI. Also draw IL parallel to GK. Since $BI=IK$, $BL=LG$ and $GI=BI=IK=GO$. By reason of the isosceles triangle BIG, we have the angle $GIO=2\angle GBO=2\angle BOA$. But $\angle GIO=\angle IOG$. $\therefore 2\angle IOA=\angle IOG$, or $IOA=\frac{1}{3}\angle HOA$.
Q. E. F.

3. THE OVAL OF CASSINI OR CASSINIAN.

1. ***The Oval of Cassini*** is the locus of the vertex of the triangle whose base is $2a$ and the product of the other sides $=m^2$.

2. $\left\{y^2+(a+x)^2\right\}\left\{y^2+(a-x)^2\right\}=m^4$ or $(x^2+y^2+a^2)^2-4a^2x^2=m^4$ is the rectangular equation of the curve, in which $2a=AB$.

3. $r^4-2a^2r^2\cos 2\theta+a^2-m^4=0$ is the polar equation of the curve.

Discussion.—If a be $>m$, there are two ovals, as shown in the figure. In that case, the last equation shows that if OPP' meets the curve in P and P', we have $OP.OP'=\sqrt{a^4-m^4}$; and therefore the curve is its own inverse with respect to a circle of radius$=\sqrt[4]{a^4-m^4}$.

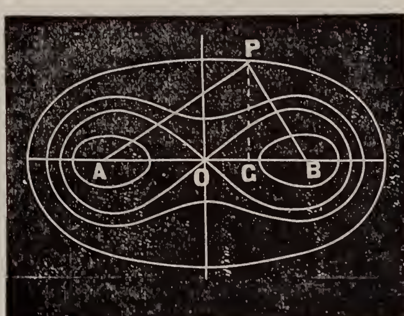

FIG. 23.

4. LEMNISCATE OF BERNOUILLI.

1. This curve is what a *cassinian* becomes when $m=a$. The above equations then reduce to

2. $(x^2+y^2)^2=2a^2(x^2-y^2)$ and

3. $r^2=2a^2\cos 2\theta$.

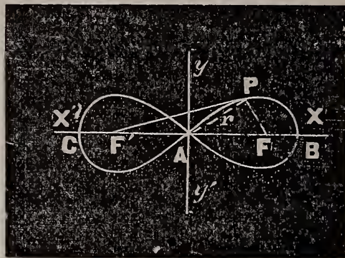

FIG. 24.

Prob. XLVII. To find the length of the arc of the Lemniscate.

Formula.—$s=\int\sqrt{1+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$$=\int\sqrt{r^2+\frac{a^4}{r^2}\left(1-\frac{r^4}{a^4}\right)}\,d\theta=\int\frac{a^2}{r}d\theta=\int_o^a-\frac{a^2dr}{\sqrt{(a^4-r^4)}}=$$

$$-a^2\int_o^a\left[\frac{1}{a^2}+\tfrac{1}{2}\cdot\frac{r^4}{a^6}+\tfrac{1}{2}\cdot\tfrac{3}{4}\frac{r^8}{a^{10}}+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\frac{r^{12}}{a^{14}}+\&c.\right]dr=a^2\left\{\frac{r}{a^2}+\tfrac{1}{2}\cdot\tfrac{1}{5}\frac{r^5}{a^6}\right.$$

$$\left.+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{9}\cdot\frac{r^9}{a^{10}}+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{1}{13}\frac{r^{13}}{a^{14}}+\&c.\right\}$$

. When $r=a$, $s=a\left[1+\tfrac{1}{2}\cdot\tfrac{1}{5}+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{9}+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{1}{13}+\&c\right]=$ arc BPA. $\therefore$ The entire length of the curve is $4a[1+\tfrac{1}{2}\cdot\tfrac{1}{5}+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{9}+\&c]$

Prob. XLVIII. To find the area of the lemniscate.

Formula.—$A=4\int\tfrac{1}{2}r^2d\theta=4a^2\int_o^{\frac{1}{4}\pi}\cos 2\theta\,d\theta=$

$$\left[2a^2\sin 2\theta\right]_o^{\frac{1}{4}\pi}=2a^2.$$

5. THE VERSIERA OR WITCH OF AGNESI.

1. ***The Versiera*** is the locus of the extremity of an ordinate to a circle, produced until the produced ordinate is to the ordinate itself, as the diameter of a circle is to one of the segments into which the ordinate divides the diameter, these segments being all taken on the same side.

2. Let P be any point of the curve, $PD=y$, the ordinate of the point P and $OD=x$, the abscissa. Then, by definition, $EP:EF::AO:EO$, or $x:EF::2a:y$. But $EF=\sqrt{AE\times EO}=\sqrt{(2a-y)y}$. $\therefore x:\sqrt{(2a-y)y}::2a:y$. Whence $x^2y=4a^2(2a-y)$ is the equation referred to rectangular co-ordinates.

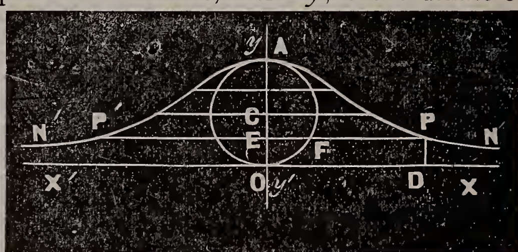

FIG. 25.

3. $r(r^2 - r^2 \sin^2\theta + 4a^2)\sin\theta = 8a^3$ is the polar equation of the curve.

Prob. XLIX. To find the length of an arc of the Versiera.

$$\textit{Formula.}—s=\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int\sqrt{1+\frac{4x^2}{(x^2+4a^2)^4}}\,dx.$$

This can be integrated by series and the result obtained approximately.

Prob. L. To find the area between the curve and its asymptote.

$$\textit{Formula.}—A=2\int y\,dx=2\times 8a^3\int_o^{2a}\frac{dx}{x^2+4a^2}=$$

$$2\left[4a^2\tan\frac{x}{2a}\right]_o^{2a}=4\pi a^2.$$

Rule.—*Multiply the area of the given circle by 4.*

NOTE.—This curve was invented by an Italian lady, Dona Maria Agnesi, 1748.

6. THE LIMACON.

1. ***The Limacon*** is the locus of a point P on the radius vector OP, of a circle OFE from a fixed point, O, on the circle and at a constant distance from either side of the circle.

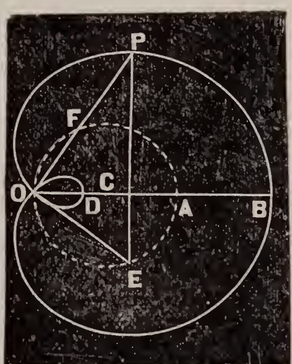

FIG. 26.

2. $(x^2+y^2-ax)^2=b^2(x^2+y^2)$ is the rectangular equation of the curve.

3. $r=a\cos\theta\pm b$ is the polar equation. In these equations, $a=OA$ and $b=PF$.

Prob. LI. To find the length of an arc of the Limacon.

$$\textit{Formula.}—s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=$$

$$\int\sqrt{(a\cos\theta+b)^2+a^2\sin^2\theta}\,d\theta=\int\sqrt{(a^2+b^2+2ab\cos\theta)}\,d\theta=$$

$$\int\sqrt{\left\{(a+b)^2\cos^2\frac{\theta}{2}+(a-b)^2\sin^2\frac{\theta}{2}\right\}}\,d\theta.$$ $\therefore$ The rectification of the Limacon depends on that of an ellipse whose semi-axes are $(a+b)$ and $(a-b.)$

When $a=b$, the curve is the *cardioid*, the polar equation of which is $r=a(1+\cos\theta)$, and $s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=$

$a\int\sqrt{2+2\cos\theta}\,d\theta=\pm2a\int\cos\frac{1}{2}\theta\,d\theta=2a\int_0^{\pi}\cos\frac{1}{2}\theta\,d\theta-$

$2a\int_{\pi}^{2\pi}\cos\frac{1}{2}\theta\,d\theta=8a=$the entire length of the cardioid.

Prob. LII. To find the area of the Limacon.

Formula.—$A=\int\frac{1}{2}r^2\,d\theta=\frac{1}{2}\int_0^{2\pi}(a\cos\theta+b)^2d\theta=$

$\pi(\frac{1}{2}a^2+b^2)$. When $a=b$, the curve becomes a cardioid, and $A=\frac{3}{2}\pi a^2$. When $a>b$, the curve has two loops and is that in the figure. $r=a\cos\theta+b$ is the polar equation of the outer loop, and $r=a\cos\theta-b$ is the polar equation of the inner loop. The area of the inner loop is $A=\int\frac{1}{2}r^2d\theta=\frac{1}{2}\int_0^{\cos^{-1}\frac{b}{a}}(a\cos\theta-b)^2d\theta=$

$(\frac{1}{2}a^2+b^2)\cos^{-1}\frac{b}{a}-\frac{3}{2}b\sqrt{a^2-b^2}$.

NOTE.—This curve was invented by Blaise Pascal in 1643. When $a=2b$, the curve is called the *Trisectrix*.

7. THE QUADRATRIX.

1. The Quadratrix is the locus of the intersection, P, of the radius. OD, and the ordinate QN, when these move uniformally, so that $ON:OA::\angle BOD:\frac{1}{2}\pi$.

2. $y=x\tan\left(\frac{a-x}{a}\cdot\frac{\pi}{2}\right)$ is the rectangular equation of the curve, in which $a=OA$, $x=ON$, and $y=IN$.

3. The curve effects the quadrature of the circle, for $OC:OB::OB:$arc ADB.

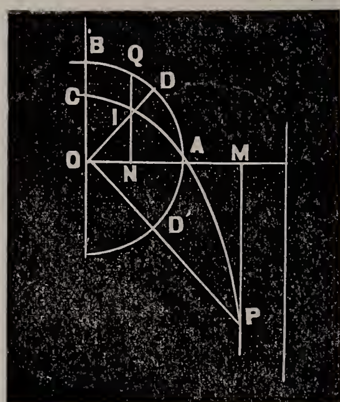

FIG. 27.

Prob. LIII. To find the area enclosed above the x—axis.

Formula.—$A=\int ydx=$

$\int x\tan\left(\frac{a-x}{a}\cdot\frac{\pi}{2}\right)dx=4a^2\pi^{-1}\log 2.$

NOTE.—This curve was invented by Dinostratus, in 370 B. C.

8. THE CATENARY.

1. ***The Catenary*** is the line which a perfectly flexible chain assumes when its ends are fastened at two points as B and C in the figure.

2. $y=\frac{1}{2}a(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$ is the rectangular equation of the curve, in which $a=OA$. A is the origin of co-ordinates. BAC is the catenary. $M'APM$ is the evolute of the catenary and is called the *Tractrix*. To find the equation of the curve, let A be the origin of co-ordinates. Let s denote the length of any arc AE; then, if p be the weight of a unit of length of the chain, the verticle tension at E, is sp. Let the horizontal tension at E, be ap, the weight of a units of length of the chain. Let EG be a tangent at E, then, if EG represents the tension of the chain at B, EF and GF will represent respectively its horizontal and its vertical tension at B.

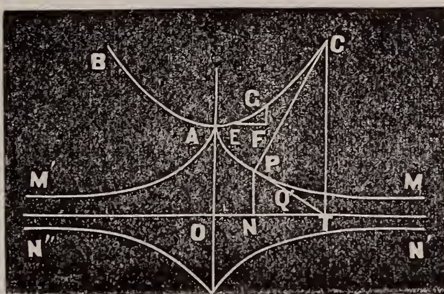

FIG. 28.

$\therefore \frac{dy}{dx}=\frac{FG}{EF}=\frac{sp}{ap}=\frac{s}{a}$. $\therefore$ $\frac{s}{a}=\frac{\sqrt{ds^2-dx^2}}{dx}$. $\therefore$ $x=a\int\frac{ds}{\sqrt{a^2+s^2}}=$ $a\log(s+\sqrt{a^2+s^2})+c$. Since $x=o$, when $s=o$, $c=-a\log a$.

$\therefore$ $x=a\log\left(\frac{s}{a}+\sqrt{1+\frac{s^2}{a^2}}\right)$. From this equation, we find $s=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ which is the length of the curve measured from A.

But $\frac{dy}{dx}=\frac{s}{a}$. $\therefore$ $\frac{dy}{dx}=\frac{s}{a}=\frac{1}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$.

$\therefore$ $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$.

Prob. LIV. To find the area of the Catenary.

Formula.—$A=\int ydx=\int\frac{1}{2}a\left\{e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right\}dx=$

$\frac{1}{2}a^2\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)=a\sqrt{y^2-a^2}$. This is the area included between the axis of x, the curve and the two ordinates, $y_1=a$, $y_2=y$.

Note.—The form of equilibrium of a flexible chain was first investigated by Galileo, who pronounced the curve to be a parabola. His error was detected experimentally, in 1669, by Joachim Jungius, a German geometer; but the true form of the Catenary was obtained by James Bernouilli, in 1691.

9. THE TRACTRIX.

1. ***The Tractrix*** is the involute of the Catenary.

2. $x=a \log \left\{ a+\sqrt{(a^2-y^2)} \right\} -a \log y-\sqrt{(a^2-y^2}$, is the rectangular equation of the curve.

Prob. LV. To find the length of an arc of the tractrix.

Formula.—$s= a \log \left(\frac{a}{y}\right)$.

Prob. LVI. To find the area included by the four branches.

Formula.—$A=\int ydx=-4\int_o^a \sqrt{a^2-y^2}\, dy=\pi a^2$.

10. THE SYNTRACTRIX.

1. ***The Syntractrix*** is the locus of a point, Q, on the tangent, PT, of the Tractrix.

2. $x= a \log \left\{ c+\sqrt{(c^2-y^2)} \right\} -a \log y-\sqrt{(c^2-y^2)}$ is the rectangular equation of the *Syntractrix*, in which c is QT, a constant length.

11. ROULETTES.

1. ***A Roulette*** is the locus of a point rigidly connected with a curve which rolls upon a fixed right line or curve.

(a) CYCLOIDS.

1. ***The Cycloid*** is the roulette generated by a point in the circumference of a circle which rolls upon a right line.

2. ***A Prolate Cycloid*** is the roulette generated by a point without the circumference of a circle which rolls upon a right line.

3. ***A Curtate Cycloid*** is the roulette generated by a point within the circumference of circle which rolls upon a right line.

4. $x=versin^{-1}y-\sqrt{2ry-y^2}$ is the rectangular equation of the cycloid referred to its base and a perpendicular at the left hand vertex. To produce this equation, let $AN=x$ and $PN=y$, P being any point of the curve. Let $OC=r$=the radius of the generatrix OPL. Now $AN=AO-NO$. But by construction AO=arc $PO=versin^{-1}FO$, or $versin^{-1}y$ to a radius r. $NO=PF=\sqrt{FL\times FO}=\sqrt{y(2r-y)}=\sqrt{2ry-y^2}$. $\therefore$ $x=versin^{-1}y-\sqrt{2ry=y^2}$ Or, we may have $x=a(\theta-sin\theta)$, and $y=a(1=cos\ \theta)$ in which θ is the angle, PCO, through which the generatrix has rolled.

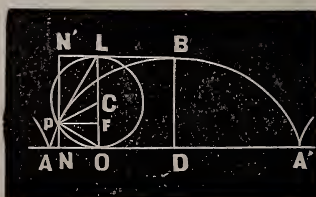

FIG. 29.

For $x=AO-NO$. But $AO=a\angle PCO=a\theta$, and $NO=PF=PC\sin\angle PCF=a\sin\theta$. $\therefore x=a\theta-a\sin\theta=a(\theta-\sin\theta)$, $y=OC-CF=a-CF$. But $CF=PC\cos\angle PCF=a\cos\theta$ $\therefore y=a-a\cos\theta=a(1-\cos\theta)$.

Prob. LVII. To find the length of an arc of the cycloid.

Formula.—$s=\int\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy=\int\sqrt{1+\frac{y^2}{2ry-y^2}}dy=$ $\sqrt{2r}\int(2r-y)^{-\frac{1}{2}}dy=-2\sqrt{2r}(2r-y)^{\frac{1}{2}}+c$. Reckoning the arc from the origin, $c=4r$; and the corrected integral is $s=-2(2r)^{\frac{1}{2}}(2r-y)^{\frac{1}{2}}+4r$. When $y=2r$, $s=4r$. $\therefore$ The whole length of the cycloid is $8r=4D$, *i e.*, the length of the cycloid is 4 times the diameter of the generating circle.

Rule.—(*1*) *Multiply the corresponding chord of the generatrix by 2. To find the length of the cycloid*:

(*2*) *Multiply the diameter of the generating circle by 4.*

I. Through what distance will a rivet in the tire of a 3-ft. buggy wheel pass in three revolutions of the wheel?

By formula, $s=3(8r)=24\times1\frac{1}{2}$ft.$=36$ft.

II. 1. 3ft.=the diameter of the wheel. Then
2. 12ft.$=4\times3$ft.=distance through which it moves in 1 revolution.
3. $\therefore$ 36ft.$=3\times12$ft.=distance through which it moves in 3 revolutions.

III. $\therefore$ It will will move through a distance of 36ft.

Prob. LVIII. To find the area of a cycloid.

Formula.—$A=2\int ydx=2\int_0^{2r}\frac{y^2dy}{\sqrt{2ry-y^2}}=$ $3r^2\text{versin}^{-1}2=3\pi r^2$.

Rule.—*Multiply the area of the generating circle by 3.*

I. What is the area of a cycloid generated by a circle whose radius is 2ft.?

By formula, $A=3\pi r^2=3\pi2^2=12\pi=37.6992$ sq. ft.

II. 1. 2ft.=the radius of the generating circle.
2. $\pi2^2=12.5664$ sq. ft.=the area of the generating circle.
3. $3\pi2^2=37.6992$ sq. ft.=the area of the cycloid,

III. $\therefore$ The area of the cycloid is 37.6992 sq. ft.

Prob. LIX. A wheel whose radius is r rolls along a horizontal line with a velocity v′; required the velocity of any point, P, in its circumference; also the velocity of P horizontally and vertically.

Since a point in the circumference of a wheel describes, in space, a cycloid, let P, Fig. 29, be the point, referred to the axes AA' and a perpendicular at A. Let (x,y) be the coordinates of the point; then will the horizontal and vertical velocities of P be the rates of change of x and y respectively. O being the point of contact, $AO=r\,versin^{-1}\frac{y}{r}$. Since the center C, is vertically over O, its velocity is equal to the rate of increase of AO. In an element of time, dt, the center C will move the distance $d\left(r\,versin^{-1}\frac{y}{r}\right)=\frac{rdy}{\sqrt{2ry-y^2}}$. $\therefore$ Its velocity $v'=$ the distance it moves divided by the time it moves, or $v'=\frac{rdy}{\sqrt{2ry-y^2}}\div dt=\frac{r}{\sqrt{2ry-y^2}}\frac{dy}{dt}$. $\therefore\ \frac{dy}{dt}=\frac{\sqrt{2ry-y^2}}{r}v'=$ the velocity vertically.... (1).

From the equation of the cycloid, $x=r\,versin^{-1}\frac{y}{r}-\sqrt{2ry-y^2}$, we have $dx=\frac{y}{\sqrt{2ry-y^2}}dy$. Now $dx\div dt=$the velocity of the point horizontally. But $dx\div dt$, or $\frac{dx}{dt},=\frac{y}{\sqrt{2ry-y^2}}\frac{dy}{dt}$. Substituting the value of $\frac{dy}{dt}$, we have $\frac{dx}{dt}=\frac{y}{r}v'$ ..... (2). An element of the curve $APBA'$ is ds and this is the distance the point travels in an element of time, dt. $\therefore\ \frac{ds}{dt}=$ the velocity of the point, P. But $ds=\sqrt{dy^2+dx^2}=\sqrt{\left(\frac{2ry-y^2}{r^2}+\frac{y^2}{r^2}\right)}\,v'dt=\sqrt{\frac{2y}{r}}\,v'dt$, since, from (1), $dy=\frac{\sqrt{2ry-y^2}}{r}\frac{v'}{dt}$ and, from (2), $dx=\left(\frac{y}{r}\cdot\frac{v'}{dt}\right)$. $\therefore$ By dividing by dt, we have $\frac{ds}{dt}=v=\sqrt{\frac{2y}{r}}\,v'=$ the velocity of the point, P...... (3). From (1), (2), and (3), we have,

if $y=0.\ \frac{dy}{dt}=0$, $\frac{dx}{dt}=0$, and $\frac{ds}{dt}=0$;

if $y=r$, $\frac{dy}{dt}=v'$, $\frac{dx}{dt}=v'$, and $\frac{ds}{dt}=\sqrt{2v'}$;

if $y=2r$, $\frac{dy}{dt}=0$, $\frac{dx}{dt}=v'$, and $\frac{ds}{dt}=2v'$.

Hence, when a point of the circumference is in contact with the line, its velocity is 0; when it is in the same horizontal plane as the center, its velocity horizontally and verically is the same as the velocity of the center, and when it is at the highest point, its motion is entirely horizontal, and its velocity is twice that of the center. Since $\frac{ds}{dt}=\sqrt{\frac{2y}{r}}v'=\frac{\sqrt{2ry}}{r}v'$, we have by proportion, $\frac{ds}{dt}:v'::\sqrt{2ry}:r$. But $\sqrt{2ry}=\sqrt{(PF^2+FO^2)}=PO$.

$\therefore$ The velocity of P is to that of C as the chord PO is to the radius CO; that is, F and C are momentarily moving about O with equal angular velocity.

(*b*) THE PROLATE AND CURTATE CYCLOID.

1. $x=a(\theta-m\sin\theta)$, $y=a(1-m\cos\theta)$ are the equations in every case.

2. The cycloid is prolate when m is >1 as $AIP'B'I'A'$, Fig. 36, and curtate when m is <1, as PB. These equations are found thus: Let $CP=ma$, and $\angle OCP=\theta$. Then $x=AN=AO-ON$. But $AO=$arc subtended by $\angle OCP=a\theta$, and $ON=PC\times\sin\angle NPC=ma\sin\angle NPC\ (=\angle PCL=\pi-\theta)=ma\sin(\pi-\theta)=ma\sin\theta$. $\therefore x=a\theta-ma\sin\theta=a(\theta-m\sin\theta)$. $y=PN=OC+PC\cos\angle NPC\ (=\angle PCL=\pi-\theta)=a+ma\cos(\pi-\theta)=a-ma\cos\theta=a(1-m\cos\theta)$. The same reasoning applies when we assume the point to be P'.

NOTE.—These curves are also called *Trochoids*.

Prob. LX. To find the length of a Trochoid.

Formula.—$s=\int\sqrt{dx^2+dy^2}$.

Since $x=a(\theta-m\sin\theta)$, $dx=a(1-m\cos\theta)d\theta$; and since $y=a(1-m\cos\theta)$, $dy=am\sin\theta d\theta$. $\therefore s=\int\sqrt{dx^2+dy^2}=a\int_0^\pi\sqrt{\left[(1-m\cos\theta)^2+m^2\sin^2\theta\right]}d\theta=a\int_0^\pi\sqrt{(1+m^2-2m\cos\theta)}d\theta=$
$4a\int_0^{\frac{1}{2}\pi}\sqrt{\left\{(1+m)^2-4m\cos^2\varphi\right\}}d\varphi=4a(1+m)\int_0^{\frac{1}{2}\pi}\sqrt{(1-\frac{4m}{(1+m)^2}\cos^2\varphi)}d\varphi$.

I. If a fly is on the spoke of a carriage wheel 5 feet in diameter, 6 inches up from the ground, through what distance will the

fly move while the wheel makes **one revolution on a level plane?**

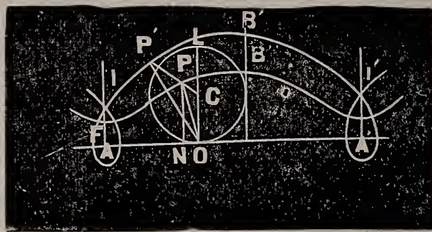

FIG. 30.

Let C be the center of the wheel, in the figure, and P the position of the fly at any time. Let OC= the radius of the carriage wheel $=a=2\frac{1}{2}$ ft., PC= 2 ft., and the angle $OCP=\theta$. Let (x,y) be the co-ordinates of the point P. Let F, a point at the intersection of the curve and AI be the position of the fly when the motion of the wheel commenced. Then since $x=a(\theta-m\sin\theta)$ and $y=a(1-m\cos\theta)$, we have $dx=a(1-m\cos\theta)d\theta$, and $dy=am\sin\theta\, d\theta$. $\therefore$ $s=\int\sqrt{(dx^2+dy^2)}=\int_0^{\pi}\sqrt{\left\{a^2(1-m\cos\theta)^2+a^2m^2\sin^2\theta\right\}}\,d\theta=a\int_0^{\pi}\sqrt{(1+m^2-2m\cos\theta)}d\theta=4a\int_0^{\frac{1}{2}\pi}\sqrt{\left\{(1+m)^2-4m\cos^2\varphi\right\}}\,d\varphi$, in which $\varphi=\frac{1}{2}\theta$. But $PC=2$ ft., and since $PC=ma=m\times2\frac{1}{2}$ ft., $m=2\div2\frac{1}{2}=\frac{4}{5}$. $\therefore$ $s=4\times2\frac{1}{2}\int_0^{\frac{1}{2}\pi}\sqrt{\left\{(1+\frac{4}{5})^2-4\times\frac{4}{5}\cos^2\varphi\right\}}\,d\varphi=10\int_0^{\frac{1}{2}\pi}\frac{1}{5}\sqrt{(81-80\cos^2\varphi)}d\varphi=$
$*\,18\int_0^{\frac{1}{2}\pi}\sqrt{(1-\frac{80}{81}\cos^2\varphi)}d\varphi=9\pi\left\{1-\frac{20}{81}-3(\frac{1}{1\cdot2})^2(\frac{20}{81})^2-5(\frac{1\cdot3}{1\cdot2\cdot3})^2(\frac{20}{81})^3-7(\frac{1\cdot3\cdot5}{1\cdot2\cdot3\cdot4})^3(\frac{20}{81})^4-\&c\right\}=18.84$ ft.

II. $\therefore$ The fly will move 18.84 ft.

Prob. LXI. To find the area contained between the trochoid and its axis.

Formula.—$A=\int ydx=2a^2\int_0^{\pi}(1-m\cos\theta)(1-m\cos\theta)d\theta=2a^2\int_0^{\pi}(1-m\cos\theta)^2d\theta=2a^2\Big(\theta-2m\sin\theta+\frac{1}{2}m^2(\theta-\sin\theta\cos\theta)\Big)_0^{\pi}=2a^2(\pi+\frac{1}{2}m^2\pi)$. When $m=1$, the curve is the cycloid and the area $=3\pi a^2$ as it should be.

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* When φ is replaced by $(\frac{1}{2}\pi+\varphi)$, this is an *elliptic integral* of the second kind and may be written $4aE(\frac{80}{81},\varphi)$.

(c) EPITROCHOID AND HYPOTROCHOID.

1. ***An Epitrochoid*** is the roulette formed by a circle rolling upon the convex circumference of a fixed circle, and carrying a generating point either within or without the rolling circle.

2. ***An Hypotrochoid*** is the roulette formed by a circle rolling upon the concave circumference of a fixed circle, and carrying a generating point either within or without the rolling circle.

3. $x=(a+b)\cos\theta - mb\cos\frac{a+b}{b}\theta$, $y=(a+b)\sin\theta - mb\ \cos\frac{a+b}{b}\theta$

are the equations of the epitrochoids.

In the figure, let C be the center of the fixed circle and O the center of the rolling circle. Let $FP'Q$ be a portion of the curve generated by the point P' situated within the rolling circle, and let $CG=x$ and $P'G=y$ be the co-ordinates of the point, P'.

Let A be the position of P when the rolling commences, and $\varphi=\angle POC$ through which it rolled. Draw OK perpendicular to CG and $P'I$ perpendicular to OK; draw DP and DP'. Let $OP'=mOP=mb$ and the angle $ACD=\theta$. Then $x=CG=CK+KG=KC+P'I$. But $CK=OC\cos\theta=(a+b)\cos\theta$ and $P'I=P'O\cos\angle OP'I=mb\cos\{\pi-(\varphi+\theta)\}=-mb\cos(\varphi+\theta)$. But $arc\ AD=arc\,PD$. $\therefore\ a\theta=b\varphi$. Whence $\varphi=\frac{a}{b}\theta$.

$\therefore\ P'I=-mb\cos\frac{a+b}{b}\theta$, and $x=(a+b)\cos\theta - mb\cos\frac{a+b}{b}\theta$. $y=P'G=IK=OK-OI$. But $OK=OC\sin\angle KCO=(a+b)\sin\theta$, and $OI=OP'\sin\angle OP'I=mb\sin\{\pi-(\varphi+\theta)\}=mb\sin(\varphi+\theta)=mb\sin\frac{a+b}{b}\theta$. $\therefore\ y=(a+b)\sin\theta - mb\sin\frac{a+b}{b}\theta$.

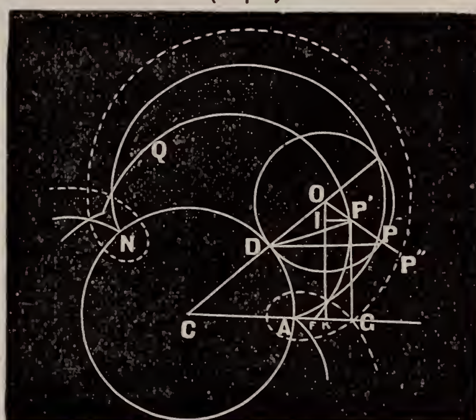

FIG. 31.

If $m=1$, the point P' will be on the circumference of the rolling circle and will describe the curve APN which is called the

Epicycloid The equations for the Epicycloid are $x=(a+b)\cos\theta-b\cos\frac{a+b}{b}\theta$, and $y=(a+b)\sin\theta-b\sin\frac{a+b}{b}\theta$. The equations for the *Hypotrochoid* may be obtained by changing the signs of b and mb, in the equations for the Epitrochoid. $\therefore$ $x=(a-b)\cos\theta+mb\cos\frac{a-b}{b}\theta$, and $y=(a-b)\sin\theta-mb\sin\frac{a-b}{b}\theta$ are the equations for the Hypotrochoid. If $m=1$, the generating point is in the circumference of the rolling circle and the curve generated will be a *Hypocycloid*. $\therefore$ $x=(a-b)\cos\theta+b\cos\frac{a-b}{b}\theta$, and $y=(a-b)\sin\theta-b\sin\frac{a-b}{b}\theta$ are the equations of the *Hypocycloid*.

Prob. LXII. To find the length of the arc of an epitrochoid.

Formula.—$s=\int\sqrt{dx^2+dy^2}=\int\sqrt{\left\{\left[-(a+b)\sin\theta+m(a+b)\sin\frac{a+b}{b}\theta\right]^2+\left[(a+b)\cos\theta-m(a+b)\cos\frac{a+b}{b}\theta\right]^2\right\}}d\theta$
$=(a+b)\int\sqrt{\left\{1+m^2-2m(\sin\theta\sin\frac{a+b}{b}\theta+\cos\theta\cos\frac{a+b}{b}\theta)\right\}}d\theta=(a+b)\int\sqrt{(1+m^2-2m\cos\frac{a}{b}\theta)}d\theta$. This may be expressed as an *elliptic integral*, $E(k,\varphi)$, of the second kind, by substituting $(\pi+\frac{2b}{a}\varphi)$ for θ, and then reducing.

2. By making $m=1$, we have $s=(a+b)\sqrt{2}\int\sqrt{(1-\cos\frac{a}{b}\theta)}\,d\theta$, the length of the arc of an hypocycloid.

3. By changing sign of b, the above formula reduces to $s=(a-b)\int\sqrt{(1+m^2+2m\cos\frac{a}{b}\theta})d\theta$, which is the length of the arc of an *hypotrochoid*.

4. By making $m=1$, in the last formula, we have $s=(a-b)\sqrt{2}\int(1+\cos\frac{a}{b}\theta)^{\frac{1}{2}}d\theta$, which is the length of the arc of an *hypocycloid*.

I. A circle 2 ft. in diameter rolls upon the convex circumference of a circle whose diameter is 6 feet. What is the length

of the curve described by a point 4 inches from the center of the rolling circle, the rolling circle having made a complete revolution about the fixed circle?

In Fig. 31, let O be the center of the rolling circle; C the center of the fixed circle; $CD=3$ ft.$=a$, the radius of fixed circle; $OD=1$ ft.$=b$, the radius of the rolling circle; $OP=4$ inches$=\frac{1}{3}$ of 12 inches$=mb$ the distance of the point from the center; and P the position of the point at any time after the rolling begins. Let $\theta=$ the angle ACD and $\varphi=$the angle POD through which the rolling circle has rolled. Then we have, as previously shown, the equations of the locus P,

$$x=(a+b)\cos\theta-mb\cos(\varphi+\theta)=(a+b)\cos\theta-mb\cos\frac{a+b}{b}\theta,$$

$$y=(a+b)\sin\theta-mb\sin(\varphi+\theta)=(a+b)\sin\theta-mb\sin\frac{a+b}{b}\theta.$$

From these equations, we can find dx and dy.

$\therefore$ By formula, $s=\int\sqrt{dx^2+dy^2}=6(a+b)\int_0^{\frac{1}{3}\pi}\sqrt{(1+m^2-}$

$2m\cos\frac{a}{b}\theta)d\theta=24\int_0^{\frac{1}{3}\pi}\sqrt{(1+(\frac{1}{3})^2-\frac{2}{3}\cos 3\theta)}d\theta=8\int_0^{\frac{1}{3}\pi}\sqrt{(10-}$

$6\cos 3\theta)d\theta$. Let $3\theta=2\psi$. Then $s=8\int_0^{\frac{1}{3}\pi}\sqrt{(10-6\cos 3\theta)}d\theta=$

$21\frac{1}{3}\int_0^{\frac{1}{2}\pi}\sqrt{(1-\frac{3}{4}\cos^2\psi)}d\psi$, $=21\frac{1}{3}\int_0^{\frac{1}{2}\pi}\Big[1-\frac{1}{2}\cdot\frac{3}{4}\cos^2\psi-\frac{1}{2}\cdot\frac{1}{4}\cdot(\frac{3}{4})^2$

$\cos^4\psi-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}(\frac{3}{4})^3\cos^6\psi-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\cdot\frac{5}{8}(\frac{3}{4})^4\cos^8\psi-\text{\&c.}\Big]d\psi=$

$21\frac{1}{3}\Big\{\psi-\frac{1}{2}\cdot\frac{3}{4}[\frac{1}{2}(\frac{1}{2}\sin 2\psi+\psi)]-\frac{1}{2}\cdot\frac{1}{4}\cdot(\frac{3}{4})^2[\frac{1}{8}(\frac{1}{4}\sin 4\psi+2\sin 2\psi+3\psi)]$

$-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}(\frac{3}{4})^3[\frac{1}{32}(\frac{1}{6}\sin 6\psi+\frac{3}{2}\sin 4\psi+\frac{15}{2}\sin 2\psi+10\psi)]-\text{\&c.}\Big\}_0^{\frac{1}{2}\pi},=$

$10\frac{2}{3}\pi\Big\{1-(\frac{1}{2})^2(\frac{3}{4})-\frac{1}{3}(\frac{1\cdot 3}{2\cdot 4})^2(\frac{3}{4})^2-\frac{1}{5}(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6})^2(\frac{3}{4})^3-\frac{1}{7}(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8})^2$

$(\frac{3}{4})^4-\text{\&c.}\Big\}=10\frac{2}{3}\pi\times.773=26.9$ ft., nearly.

Remark.—When the point is on the circumference of the rolling wheel, the length of the curve generated by the point is $s=$

$(a+b)\int\sqrt{(1+m^2-2m\cos\frac{a}{b}\theta)}d\theta=(a+b)\int\sqrt{(1+1-2\cos\frac{a}{b}\theta)}d\theta$.

If we let the conditions of the above problem remain the same,

only changing the generating point to the circumference, we have for the length of the curve, $s=6\sqrt{2}(3+1)\int_0^{\frac{1}{3}\pi}\sqrt{(1-\cos 3\theta)}d\theta=$ $48\int_0^{\frac{1}{2}\pi}\sqrt{(1-\frac{1}{2}\cos^2\varphi)}d\varphi$, where $\varphi=\frac{3}{2}\theta$. Expanding this by the Binomial Theorem and integrating each term separately, $s=$ $24\pi\left\{1-(\frac{1}{2})^2(\frac{1}{2})-\frac{1}{3}(\frac{1}{2}\cdot\frac{3}{4})^2(\frac{1}{2})^2-\frac{1}{5}(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6})^2(\frac{1}{2})^3-\text{\&c.}\right\}$

I. A circle whose radius is 1 ft. is rolled on the concave circumference of a circle whose radius is 4 ft. What is the length of the curve generated by a point in the circumference of the rolling circle, the rolling circle having returned to the point of starting?

$x=(a-b)\cos\theta+b\cos\dfrac{a-b}{b}\theta$,

$y=(a-b)\sin\theta-b\sin\dfrac{a-b}{b}\theta$, are the equations of the curve which is a hypocycloid. In these equations $a=4$ and $b=1$.

$\therefore\ x=3\cos\theta+\cos 3\theta=4\cos^3\theta$, and

$y=3\sin\theta-\sin 3\theta=4\sin^3\theta$. Whence,

$\cos\theta=\sqrt[3]{\left(\dfrac{x}{4}\right)}$, and $\sin\theta=\sqrt[3]{\left(\dfrac{y}{4}\right)}$.

$\therefore\ \cos^2\theta+\sin^2\theta=\left(\dfrac{x}{4}\right)^{\frac{2}{3}}+\left(\dfrac{y}{4}\right)^{\frac{2}{3}}$. But $\cos^2\theta+\sin^2\theta=1$.

$\therefore\ \left(\dfrac{x}{4}\right)^{\frac{2}{3}}+\left(\dfrac{y}{4}\right)^{\frac{2}{3}}=1$, Whence,

$x^{\frac{2}{3}}+y^{\frac{2}{3}}=4^{\frac{2}{3}}$, which is the rectangular equation of the curve.

$\therefore$ By formula, $s=\sqrt{(dx^2+dy^2)}=4\int_0^a\left(\dfrac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right)^{\frac{1}{2}}dx=$

$4a^{\frac{1}{3}}\int_0^a x^{-\frac{1}{3}}dx=4a^{\frac{1}{3}}\left[\frac{3}{2}x^{\frac{2}{3}}\right]_0^a=6a=6\times4=24$ ft

X. PLANE SPIRALS.

1. ***A Plane Spiral*** is the locus of a point revolving about a fixed point and continually receding from it in such a manner that the radius vector is a function of the variable angle. Such a curve may cut a right line in an infinite number of points. This would render its rectilinear equation of an infinite degree. Hence, these loci are *transcendental.*

2. ***The Measuring Circle*** is the circle whose radius is the radius vector of the spiral, at the end of one revolution of the generating point in the positive direction.

3. ***A Spire*** is the portion of the spiral generated by any one revolution of the generating point.

1. THE SPIRAL OF ARCHIMEDES.

1. ***The Spiral of Archimedes*** is the locus of a point revolving about and receding from a fixed point so that the ratio of the radius vector to the angle through which it has moved from the polar axis, is constant.

2. $r=a\theta$ is the polar equation of this curve.

Prob. LXIII. To find the length of the spiral of Archimedes.

Formula.—$s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\int\sqrt{(r^2+a^2)}\,d\theta=a\int\sqrt{(1+\theta^2)}\,d\theta=\frac{1}{2}a\theta\sqrt{(1+\theta^2)}+\frac{1}{2}a\log\left\{\theta+\sqrt{(1+\theta^2)}\right\}$, which is the length of the curve measured from the origin.

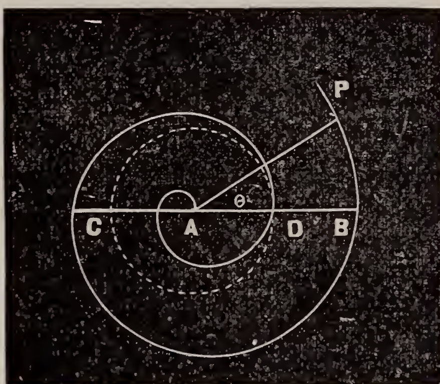

FIG. 32.

$s=a\pi\sqrt{[1+(2\pi)^2]}+\frac{1}{2}a\log\left\{2\pi+\sqrt{[1+(2\pi)^2]}\right\}$ is the length of the curve made by one revolution of the generating point.

Prob. LXIV. To find the area of the spiral of Archimedes.

Formula.—$A=\frac{1}{2}\int r^2 d\theta=\frac{1}{2}a^2\int\theta^2 d\theta=\frac{1}{6}a^2\theta^3=\frac{r^3}{6a}$, the area measured from the origin.

2. THE RECIPROCAL OR HYPERBOLIC SPIRAL.

1. ***The Reciprocal or Hyperbolic Spiral*** is the locus of a point revolving around and receding from a fixed point so that the inverse ratio of the radius vector to the angle through which it has moved from the polar axis, is constant.

2. $r=\frac{a}{\theta}$ is the polar equation of the *Hyperbolic Spiral.*

Prob. LXV. To find the length of the Hyperbolic Spiral.

Formula.—$s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\frac{a}{\theta^2}\int\sqrt{(1+\theta^2)}\,d\theta$

$=\theta\sqrt{(1+\theta^2)}+\log\left\{\theta+\sqrt{(1+\theta^2)}\right\}-$

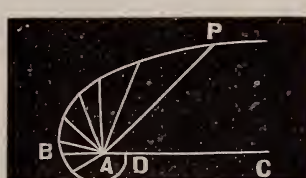

FIG. 33.

$\theta^{-1}(1+\theta^2)^{\frac{3}{2}}=\log\left\{+\theta\sqrt{(1+\theta^2)}\right\}-\theta^{-1}\sqrt{1+\theta^2}$, is the length of the spiral measured from the origin.

Prob. LXVI. To find the area of the Hyperbolic Spiral.

Formula.—$A=\frac{1}{2}\int r^2\,d\theta=\frac{1}{2}a^2\int\frac{d\theta}{\theta^2}=-\frac{a^2}{2\theta}$, the area measured from the origin. This result must be made positive since the radius vector revolves in the negative direction.

3. THE LITUUS.

1. ***The Lituus*** is the locus of a point revolving around and receding from a fixed point so that the inverse ratio of the radius vector to the square root of the angle through which it has moved, is constant.

2. $r=\frac{a}{\sqrt{\theta}}$ is the equation of the Lituus.

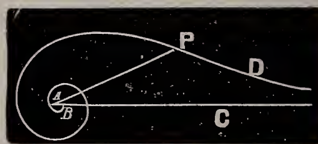

FIG. 34.

Prob. LXVII. To find the length of the Lituus.

Formula.—$s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\frac{1}{2}a\theta^{-\frac{3}{2}}\int\sqrt{(1+4\theta^2)}\,d.\theta=$

$$\left[-\tfrac{1}{8}a\left\{\theta^{-\frac{1}{2}}(1+\theta^2)^{\frac{3}{2}}\right\}-\tfrac{5}{16}\theta^{\frac{3}{2}}\left(\tfrac{2}{3}-\tfrac{1}{7}\theta^2-\tfrac{1}{44}\theta^4-\tfrac{1}{120}\theta^6-\&c.\right]_{\theta'}^{\theta''}.$$

Prob. LXVIII. To find the area of the Lituus.

Formula.—$A=\frac{1}{2}\int r^2\,d\theta=\frac{a^2}{2}\int\frac{d\theta}{\theta}=\frac{1}{2}a^2\log\theta.$

4. THE LOGARITHMIC SPIRAL.

1. ***The Logarithmic Spiral*** is the locus generated by a point revolving around and receding from a fixed point in such a manner that the radius vector increases in a geometrical ratio, while the variable angle increases in an arithmetrical ratio.

2. $r=a^\theta$ is the polar equation of the Logarithmic Spiral. If a is the base of a system of logarithms, this equation becomes $\theta=\log r$.

Prob. LXIX. To find the length of the Logarithmic Spiral.

Formula.—$s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\int\sqrt{r^2+\left(\frac{r^2}{m^2}\right)}d\theta=$

$(m^2+1)^{\frac{1}{2}}dr=\sqrt{(m^2+1)}r$, where m is the modulus of the system of logarithms.

Prob. LXX. To find the area of the Logarithmic Spiral.

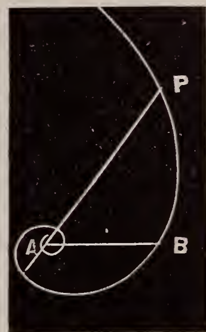

FIG. 35.

Formula.—$A=\frac{1}{2}\int r^2 d\theta=\frac{m}{2}\int r dr=\frac{1}{4}mr^2$. Since $m=1$, in the Naparian System of Logarithms, $A=\frac{1}{4}r^2$, *i. e.*, the area is $\frac{1}{4}$ of the square of the radius vector.

XI. MENSURATION OF SOLIDS.

Prob. LXXI. To find the solidity of a cube, the length of its edge being given.

Formula.— $V=(\text{edge})\times(\text{edge})\times(\text{edge})=(\text{edge})^3$.

Rule.—*Multiply the edge of the cube by itself, and that product again by the edge.*

I. What is the volume of a cube whose edge is 5 feet?

By formula, $V=(\text{edge})^3=(5)^3=125$ cu. ft.

II. $\begin{cases} 1.\ 5 \text{ ft.}=\text{the edge of the cube.} \\ 2.\ 5\times5\times5=125 \text{ cu. ft.}=\text{the volume of the cube.} \end{cases}$

III. ∴ The volume of the cube is 125 cu. ft.

Remark.—Some teachers of mathematics prefer to express the volume by saying $5\times5\times5\times1$ cu. ft. $=125\times1$ cu. ft. $=125$ cu. ft.

Prob. LXXII. To find the volume of a cube, having given its diagonal.

Formula.— $V=\left(\frac{d}{\sqrt{3}}\right)^3$

Rule.—*Divide the diagonal by the square root of 3, and the cube of the quotient will be the volume of the cube.*

What is the volume of a cube whose diagonal is 51.9615 inches?

By formula, $V=\left(\frac{d}{\sqrt{3}}\right)^3=\left(\frac{51.9615}{\sqrt{3}}\right)^3=\left(\frac{51.9615}{1.73205}\right)^3=$ 27,000 cu. in.

II.
1. 51.9615 in.=the diagonal.
2. 30 in.=51.9615 in.$\div\sqrt{3}$=51.9615 in.$\div$1.73205=the edge of the cube.
3. ∴ 30×30×30=27,000 cu. in.=the volume of the cube.

III. ∴ The volume of the cube whose diagonal is 51.9615 in., is 27,000 cu. in.

Prob. LXXIII. To find the volume of a cube whose surface is given.

Formula.—$V=\left(\sqrt{\frac{S}{6}}\right)^3$.

Rule.—*Divide the surface of the cube by 6 and extract the squae root of the quotient. This will give the edge of the cube. The cube of the edge will be the volume of the cube.*

I. What is the volume of a cube whose surface is 294 square feet?

By formula, $V=\left(\sqrt{\frac{S}{6}}\right)^3=\left(\sqrt{\frac{294}{6}}\right)^3=(\sqrt{49})^3=7^3=$ 243 cu. in.

II.
1. 294 sq. ft.=the surface of the cube.
2. 49 sq. ft.=294 sq. ft.$\div$6=area of one side of the cube.
3. $\sqrt{49}$=7 ft.=length of the edge of the cube.
4. ∴ 7×7×7=343 cu. ft.=volume of cube.

III. ∴ 343 cu. ft. is the volume of a cube whose surface is 294 sq. ft.

Prob. LXXIV. To find the solidity of a parallelopipedon.

Formula.—$V=l\times b\times t$, where l=length, b=breadth, and t=thickness.

Rule.—*Multiply the length, breadth and thickness together.*

I. What is the volume of a parallelopipedon whose length is 24 feet, breadth 8 feet, and thickness 5 feet?

By formula, $V=l\times b\times t=24\times8\times5=960$ cu. ft.

II.
1. 24 ft.=the length.
2. 8 ft.=the breadth, and
3. 5 ft.=the thickness.
4. ∴ 24×8×5=960 cu. ft.=the volume.

III. ∴ 960 cu. ft.=the length of the parallelopipedon.

Prob. LXXV. To find the dimentions of a parallelopipedon, having given the ratio of its dimensions and the volume.

Formula.—$l=\sqrt[3]{[V\div(m\times n\times p)]}m$; $b=\sqrt[3]{[V\div(m\times n\times p)]}n$; and $t=\sqrt[3]{[V\div(m\times n\times p)]}p$, where m, n, and p are the ratios of the length, breadth, and thickness respectively.

Rule.—*Divide the volume of the parallelopipedon by the product of the ratios of the dimensions, and extract the the cube root of the quotient. This gives the G. C. D. of the three dimensions. Multiply the ratios of the dimensions by the G. C. D., and the results will be the dimensions respetively.*

I. What are the dimensions of a parallelopipedon whose length, breadth and thickness are in the ratios of 5, 4 and 3; and whose volume is 12960 cu. ft.?

By formula, $l=\sqrt[3]{[12960\div(5\times4\times3)]}5=30$ ft.; $b=\sqrt[3]{[12960\div(5\times4\times3)]}4=24$ ft.; and $t=\sqrt[3]{[12960\div(5\times4\times3)]}3=18$ ft.

II.
1. 5=the quotient obtained by dividing the length by the G. C. D. of the three dimensions.
2. 4=the quotient obtained by dividing the breadth by the G. C. D. of the three dimensions.
3. 3=the quotient obtained by dividing the thickness by the G. C. D. of the three dimensions.
4. ∵ $5\times$G. C. D.=the length,
5. $4\times$G. C. D.=the breadth, and
6. $3\times$G. C. D.=the thickness.
7. ∴ $(5\times$G. C. D.$)\times(4\times$G. C. D.$)\times(3\times$G. C. D.$)=60\times$(G. C. D.$)^3$=the volume of the parallelopipedon.
8. ∴ 60 (G. C. D.$)^3$=12960 cu. ft.
9. (G. C. D.$)^3=12960\div60=216$.
10. ∴ G. C. D.$=\sqrt[3]{216}=6$.
11. ∴ $5\times$(G. C. D.)$=5\times6=30$ ft.=the length,
12. $4\times$(G. C. D.)$=4\times6=24$ ft.=the breadth, and
13. $3\times$(G. C. D.)$=3\times6=18$ ft.=the thickness.

III. ∴ 30 ft., 24 ft., and 18 ft. are the dimensions of the parallelopipedon.

Prob. LXXVI. To find the convex surface of a prism.

Formula.—$S=p\times a$, in which p is the perimeter of the base and a the altitude.

Rule.—*Multiply the perimeter of the base by the altitude.*

I. What is the convex surface of the prism $ABC—D$, if the altitude AE is 12 feet, AB, 6 feet, AC, 5 feet, and BC, 4 feet.?

By formula, $S=a\times p=12\times(9+5+4)=180$ sq. ft.

II. 1. 12 ft.=the altitude of the prism.
2. 6 ft.+5 ft.+4 ft.=15 ft.=the perimeter of the base.
3. ∴ $12\times15=180$ sq. ft.=the convex surface of the prism.

III. ∴ The convex surface of the prism is 180 sq. ft.

Remark.—If the entire surface is required; to the convex surface, add the area of the two bases.

Formula.—$T=S+2A$, where $2A$ is the area of the base, S the convex surface, and T the total surface.

Prob. LXXVII. To find the volume of a prism.

Formula.—$V=a\times A$, where A is the area of the base, a, the altitude.

Rule.—*Multiply the area of the base by the altitude.*

I. What is the volume of the triangular prism $ABC—D$, whose length AE is 8 feet, and either of the equal sides AB, BC, or AC, $2\frac{1}{2}$ feet?

By formula, $V=a\times A=8\times[(2\frac{1}{2})^2\frac{1}{4}\sqrt{3}]=12\frac{1}{2}\sqrt{3}=21.6506$ cu. ft.

II. 1. 8 ft.=the altitude AE.
2. $2\frac{1}{2}$ ft.=the length of one of the equal sides of the base, as AB.
3. $(2\frac{1}{2})^2\frac{1}{4}\sqrt{3}$=the area of the base ABC, by Prob. XI.
4. ∴ $8\times(2\frac{1}{2})^2\frac{1}{4}\sqrt{3}=12\sqrt{3}=21.6506$ cu. ft. =the volume of the prism.

III. ∴ 21.6506 cu. ft.=the volume of the prism.

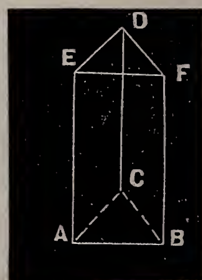

FIG. 36.

1. THE CYLINDER.

Prob. LXXVIII. To find the convex surface of a cylinder.

Formula.—$S=a\times C$, in which a is the altitude and C the circumference of the base.

Rule.—*Multiply the circumference of the base by the altitude.*

I. What is the convex surface of the right cylinder $AGB—C$, whose altitude EF is 20 feet and the diameter of its base AB is 4 feet?

By formula, $S=a\times C=20\times 4\pi=80\pi=251.32736$ sq. ft.

II.
1. 20 ft.=the altitude EF.
2. 4 ft.=the diameter AB of the base.
3. 12.566368 ft.$=4\pi=4\times 3.141592=$the circumference of the base.
4. $\therefore$ $20\times 12.566368=251.32736$ sq. ft.= the convex surface of the cylinder.

III. $\therefore$ The convex surface of the cylinder is 251.32736 sq. ft.

Remark.—If the entire surface is required; to th the convex surface, add the area of the two bases.

Fomula.—$T=S+2A=2\pi aR+2\pi R^2$.

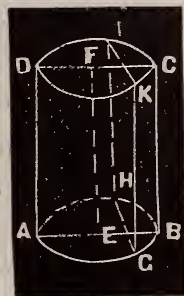

FIG. 37.

Prob. LXXIX. To find the solidity of a cylinder.

Formula.—$V=a\times A$, in which A is the area of the base.

Rule.—*Multiply the area of the base by the altitude.*

I What is the solidity of the cylinder $AGB—C$, whose altitude FE is 8 feet and diameter AB of the base 2 feet?

By formula, $V=a\times A=8\times(1^2\pi)=8\pi=8\times 3.141592=$ 25.132736 cu. ft.

II.
1. 8 ft.=the altitude, EF.
2. 2 ft.=the diameter, AB, of the base.
3. 3.141592 sq. ft.$=\pi R^2=\pi 1^2=$area of the base.
4. $\therefore$ $8\times 3.141592=25.132736$ cu. ft.

III. $\therefore$ 25.132736 cu. ft. is the volume of the cylinder.

2. CYLINDRIC UNGULAS.

1. ***A Cylindric Ungula*** is any portion of a cylinder cut off by a plane.

Prob. LXXX. To find the convex surface of a cylindric ungula, when the cutting plane is parallel to the axis of the cylinder.

Formula.—$S=a\int ds=2a\int_0^y \frac{r\,dy}{(r^2-y^2)^{\frac{1}{2}}}=2ar\sin^{-1}\frac{y}{r}=a\times$ *arc* of the base.

Rule.—*Multiply the arc of the base by the altitude.*

I. What is the surface of the cylindric ungula $API—Q$, whose altitude AD is 32 feet and height AT of the arc of the base, 2 feet and cord PI of the base 12 feet?

By formula, $S=a\times arc\ PAI=a2r\sin^{-1}\frac{y}{r}=a\times 2r\sin^{-1}\left(\frac{y}{r}\right)$

$=a\times 2\left(\frac{IT^2+AT^2}{2AT}\right)\sin^{-1}\left[IT\div\left(\frac{IT^2+AT^2}{2AT}\right)\right]=$

$32\left(\frac{6^2+2^2}{2}\right)\sin^{-1}\frac{3}{5}=640\left(\frac{17731}{86400}\pi\right)=411.84$ sq. ft., nearly.

The arc corresponding to the $\sin\frac{3}{5}$ is found from a table of natural sines and cosines to be $(36°\ 52\frac{5}{24}'\div 360°)$ of 2π or $\frac{17731}{86400}\pi$.

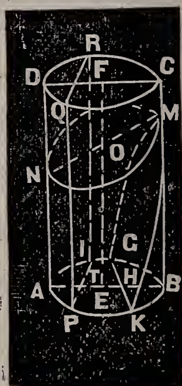

FIG. 38.

II.
1. 2 ft.=the height AT of the arc PAI.
2. 12 ft.=the length of the chord PI.
3. 12.87 ft.$=2\sqrt{6^2+2^2}\times\left(1+\frac{10\times 2^2}{60\times 6^2+33\times 2^2}\right)$=the length of the arc PAI, by Prob. XXV.
4. ∴ $32\times 12.87=411.84$ sq. ft. = convex surface $PAI-D$.

III. ∴ The convex surface of the cylindric ungula $PAI-Q$ is 411.84 sq. ft.

Remark.—r is found, by Prob. XX, formula $R=(a^2+c^2)\div 2a$.

Prob. LXXXI. To find the volume of a cylindric ungula, whose cutting plane is parallel to the axis.

Formula.—$V=2\int_0^y\int_0^{\sqrt{r^2-y^2}}\int_0^y dy\,dx\,dz-2zy(r^2-y^2)^{\frac{1}{2}}$

$=2a\int_0^y\int_0^{\sqrt{r^2-y^2}}dy\,dx-2ay(r^2-y^2)^{\frac{1}{2}}=2a\int_0^y\sqrt{(r^2-y^2)}dy-$

$2ay(r^2-y^2)^{\frac{1}{2}}=a\left\{y(r^2-y^2)^{\frac{1}{2}}+r^2\sin^{-1}\frac{y}{r}-2y(r^2-y^2)^{\frac{1}{2}}\right\}=$

$a\left\{r^2\sin^{-1}\frac{y}{r}-y(r^2-y^2)^{\frac{1}{2}}\right\}$, in which y is half the chord of the base. In this formula $\left(r^2\sin^{-1}\frac{y}{r}\right)$ is the area of the sector $APEIA$, and $y(r^2-y^2)^{\frac{1}{2}}$ is the area of the triangle PEI formed by joining the center E with P and I.

Rule.—*Multiply the area of the base by the altitude.*

I. What is the volume of the cylindric ungula $PIA-D$, if PI is 12 feet, AT 2 feet, and altitude AD 40 feet?

By formula, $V=aA=a\left\{r^2 \sin^{-1}\frac{y}{r}-y(r^2-y^2)^{\frac{1}{2}}\right\}=40\left\{10^2 \sin^{-1}\frac{3}{5}\right.$ $\left.-6(10^2-6^2)^{\frac{1}{2}}\right\}=4000 \sin^{-1}\frac{3}{5}-1920=4000\left(\frac{17731}{86400}\pi\right)-1920=$ $2574.016-1920=654.016$ cu. ft.

II.
1. 40 ft.$=$the altitude AD.
2. 2 ft.$=$the height AT of the arc of the base.
3. 12 ft.$=$the chord PI of the base.
4. $16\frac{1}{3}$ sq. ft.$=\frac{2^2}{2\times12}+\frac{2}{3}$ of$(2\times12)=$the area of the base, by rule, Prob. XXVIII.
5. $\therefore 40\times16\frac{1}{3}=653\frac{1}{3}$ cu. ft.$=$the volume of the cylindrical ungula $PIA-D$.

III. $\therefore$ $653\frac{1}{3}$ cu. ft.$=$the volume of the cylindrical ungula.

Remark.—A nearer result would have been obtained by finding the length of the arc PAI and multiplying it by half the radius. This would give the area of the sector $IEPA$. From the area of the sector subtract the area of the triangle PIE formed by joining P and I with E, and the remainder would be the area of the segment PIA.

Prob. LXXXII. To find the convex surface of a cylindric ungula, when the plane passes obliquely through the opposite sides of the cylinder.

Formula.—$S=\frac{1}{2}(a+a')2\pi r$, where a and a' are the least and greatest lengths of the ungula and $2\pi r$ the circumference of the base of the cylinder.

Rule.—*Multiply the circumference of the base by half the sum of the greatest and least lengths of the ungula.*

I. What is the convex surface of the cylindric ungula $AKBA-NM$, if AN is 8 feet, BM 12 feet and the radius BE of the base 3 feet?

By formula, $S=\frac{1}{2}(a+a')2\pi r=\pi(a+a')\,r=\pi(8+12)\times3=$ 188.49552 sq. ft.

II.
1. 8 ft.$=$the least length AN of the ungula, and
2. 12 ft.$=$the greatest length BM.
3. 10 ft.$=\frac{1}{2}$(8ft.$+$12 ft.)$=$half the sum of the least and greatest lengths.
4. 18.849552 ft.$=6\pi=$the circumferenc of the base.
5. $\therefore$ $10\times18.849552=188.49552$ sq. ft.$=$the convex surface.

III. ∴ 188.49552 sq. ft.=the convex surface of the ungula.

Prob. LXXXIII. To find the volume of a cylindric ungula, when the plane passes obliquely through the opposite sides of the cylinder.

Formula.— $V=\frac{1}{2}(a+a')\pi r^2=\frac{1}{2}\pi(a+a')r^2$.

Rule.—*Multiply the area of the base, by half the least and greatest lengths of the ungula.*

I. What is the volume of a cylindric ungula whose least length is 7 feet, greatest length 11 feet, and the radius of the base 2 feet?

By formula, $V=\frac{1}{2}(a+a')\pi r^2=\frac{1}{2}(7+11)\pi 2^2=113.097312$ cu. ft.

II.
1. 7 ft.=the least length of the ungula, and
2. 11 ft.=the greatest length.
3. 9 ft.=$\frac{1}{2}$(7 ft.+11 ft.)=half the length of the least and greatest lengths.
4. 12.566368 sq. ft.=$\pi 2^2$=the area of the base.
5. ∴ 9×12.566368=113.097312 cu. ft.=the volume of the ungula.

III. ∴ The volume of the ungula is 113.097312 cu. ft.

Prob. LXXXIV. To find the convex surface of a cylindric ungula, when the plane passes through the base and one of its sides.

$$*\textit{Formula.}\text{—}S=2\int_0^b \frac{a}{b}(b-x)ds=2\int_0^b \frac{a}{b}(b-x)\frac{rdx}{\sqrt{2rx-x^2}}$$
$$=2r\frac{a}{b}\int_0^b \frac{b-x}{\sqrt{2rx-x^2}}dx=2r\frac{a}{b}\Big[b\text{ vers}^{-1}\frac{x}{r}+\sqrt{2rx-x^2}-r\text{ vers}^{-1}\frac{x}{r}\Big]_0^b,$$
$$=2r\frac{a}{b}\Big[\sqrt{2rx-x^2}-(r-b)\text{vers}^{-1}\frac{x}{r}\Big]_0^b=2r\frac{a}{b}\Big[\sqrt{2rb-b^2}-$$
$$(r-b)\text{vers}^{-1}\frac{b}{r}\Big].$$

Rule.—*Multiply the sine of half the arc of the base by the diameter of the cylinder, and from the product subtract the product of the arc and cosine; this difference multiplied by the quotient of the height divided by the versed sine will be the convex surface.*

I. What is the convex surface of the cylindric ungula ACB—

D, whose altitude BD is 28 feet, height BM of arc of base 4 feet and chord AC 16 feet?

FIG. 39.

By formula, $S=2r\frac{a}{b}\left[\sqrt{2rb-b^2}-(r-b)\text{vers}^{-1}\frac{b}{r}\right]$, $=2\times10\times\frac{28}{4}\left[\sqrt{2\times10\times4-4^2}-(10-4)\text{vers}^{-1}\frac{4}{10}\right]$, $=140\left[8-6\text{ vers}^{-1}\frac{2}{5}\right]=140\left[8-\frac{54193}{367200}2\pi\right]$, $=140[8-5.5638]=341.068$ sq. ft.

II.
1. 28 ft.=the altitude BD.
2. 4 ft.=the height BM of the arc of the base.
3. 16 ft.=the chord AC of the arc of the base.
4. 8 ft.=the sine CM of the arc.
5. 10 ft.$=(8^2+4^2)\div(2\times4)$=the radius $OC=OB$ of the base, by Prob. XX, formula $R=(a^2+c^2)\div2a$.
6. 6 ft.=10 ft.—4 ft.=cosine OM of the arc.
7. 160 sq. ft.$=20\times8$=sine multiplied by the diameter of the base.
8. 18.5438 ft.$=2\sqrt{8^2+4^2}\left(1+\frac{10\times4^2}{60\times8^2+33\times4^2}\right)$=the arc CBA, by formula of Prob. XXV.
9. ∴ 111.2628 sq. ft.$=6\times18.5438$=the arc multiplied by the cosine OM.
10. 160 sq. ft.—111.2628 sq. ft.=48.7372 sq. ft.=the difference.
11. ∴ 341.1604 sq. ft.$=(28\div4)\times48.7372$ sq. ft.=the convex surface.

III. ∴ The convex surface is 341.1604 sq. ft. nearly.

NOTE.—The difference in the two answers is caused by the length of the arc CBA, in the solution, only being a near approximation.

**Demonstration.*—In the figure, let $BK=x$, $BM=b$, $BD=a$, and the angle $BMD=\theta$. Then $MK=b-x$, and $IK=FL=MK\tan\theta=(b-x)\tan\theta$. But $\tan\theta=\frac{BD}{BM}=\frac{a}{b}$. ∴ $FL=\frac{a}{b}(b-x)$. Now if we take an elemennt of the arc LBH, and from it draw a line parallel to FL, we will have an element of the sur-

face $LBHEGF$. This will be a rectangle whose length is $FL=\frac{a}{b}(b-x)$ and width an element of the arc LBH. An element of the arc is $ds=\sqrt{(dx^2+dy^2)}$. Let $HK=y$. Then $y^2=2rx-x^2$, by a property of the circle, from which we find $dy=\frac{r-x}{\sqrt{2rx-x^2}}dx$. $\therefore$ $ds=\frac{rdx}{\sqrt{2rx-x^2}}$. $\therefore$ The area of the element of the surface is $\frac{a}{b}(b-x)\frac{rdx}{\sqrt{2rx-x^2}}$, and the whole surface of $ABC-D$ is $S=2\int_0^b \frac{a}{b}(b-x)\frac{rdx}{\sqrt{2rx-x^2}}$, $=2r\frac{a}{b}\int_0^b (b-x)\frac{dx}{\sqrt{2rx-x^2}}=$ $2r\frac{a}{b}\left[\sqrt{2rx-x^2}-(r-b)\text{vers}^{-1}\frac{x}{r}\right]$, $=\frac{a}{b}\left[2r\sqrt{2rb-b^2}-2(r-b)r\text{ vers}^{-1}\frac{b}{r}\right]$. *Q. E. D.*

Prob, LXXXV. To find the volume of a cylindric ungula, when the cutting plane passes through the base and one of its sides.

Formula.— $V=\int_0^b (b-x)dA=\frac{a}{b}\int_0^b (b-x)2\sqrt{2rx-x^2}dx$,

$=2\frac{a}{b}\left[\frac{1}{3}(2rx-x^2)^{\frac{3}{2}}-(r-b)\int_0^b \sqrt{(2rx-x^2)}dx\right]=2\frac{a}{b}\left[\frac{1}{3}(2rx-x^2)^{\frac{3}{2}}\right.$ $\left.+\frac{1}{2}(r-x)\sqrt{2rx-x^2}+\frac{1}{2}r^2\sin^{-1}\frac{r-x}{r}+C\right]_0^b$. When $x=0$, $V=0$.

$\therefore$ $C=-\frac{1}{4}\pi r^2(r-b)$. $\therefore$ $V=\frac{a}{b}\left[\frac{2}{3}(2rb-b^2)^{\frac{3}{2}}-(r-b)\left\{\frac{1}{2}\pi r^2-(r-b)\sqrt{2rb-b^2}-r^2\sin^{-1}\frac{r-b}{r}\right\}\right]$.

Rule.—*From $\frac{2}{3}$ of the cube of half the chord of the base, subtract the product of the area of the base and the difference of the radius of the base and the height of the arc of the base; this difference multiplied by the quotient of the altitude of the ungula by the height (versed sine) of the arc of the base, will give the volume.*

I. What is the volume of a cylindric ungula, whose altitude BD is 8 feet, chord AC of base 6 feet, and height BM of arc of base 1 foot?

By formula, $V=\frac{a}{b}\left\{\frac{2}{3}(2rb-b^2)^{\frac{3}{2}}-(r-b)\left[\frac{1}{2}\pi r^2-\right.\right.$

$(r-b)\sqrt{(2rb-b^2)}-r^2\sin^{-1}\frac{r-b}{r}\Big]\Big\}=8\Big\{\frac{2}{3}(2\times5\times1-1)^{\frac{3}{2}}-$

$(5-1)\Big[\frac{1}{2}\pi5^2-4\sqrt{2\times5\times1-1}-5^2\sin^{-1}\frac{5-1}{5}\Big]\Big\},=8\Big\{18-$

$4\Big[\frac{1}{2}\pi25-12-25\sin^{-1}\frac{4}{5}\Big]\Big\}=528+800\sin^{-1}\frac{4}{5}-200\pi=$

13.20394 cu. ft.

II.
1. 8 ft=the altitude BD.
2. 1 ft.=the altitude BM of the arc ABC of the base.
3. 6 ft.=the chord AC of the base.
4. 18 cu. ft.$=\frac{2}{3}$ of $3^3=\frac{2}{3}$ of the cube of the sine of half the arc of the base.
5. $4\frac{1}{12}$ sq. ft.$=\frac{1^3}{2\times6}+\frac{2}{3}$ of 6×1=area of the base, by formula, (b), Prob. XXVIII.
6. $16\frac{1}{3}$ cu. ft.$=4\times4\frac{1}{12}$=the area of the base $\times$ OM, the cosine of the arc CHB.
7. $\therefore$ 8(18 cu. ft.—$16\frac{1}{3}$ cu. ft.)$=13\frac{1}{3}$ cu. ft.=the volume of the cylindric ungula $ACB-D$.

III. $\therefore$ The volume of the cylindric ungula $ACB-D$ is $13\frac{1}{3}$ cu. ft., *nearly*.

Prob. LXXXVI. To find the convex surface of the frustum of a cylindric ungula.

$$\textit{Formula.}—S=\frac{a}{b}\Big[2r\sqrt{2rb-b^2}-2(r-b)r\operatorname{vers}^{-1}\frac{b}{r}\Big]-$$

$$\frac{a'}{b'}\Big[2r\sqrt{2rb'-b'^2}-2(r-b')r\operatorname{vers}^{-1}\frac{b'}{r}\Big].$$

Rule.—(1) *Conceive the section to be continued, till it meets the side of the cylinder produced; then say, as the difference of the heights of the arcs of the two ends of the ungula, is to the height of the arc of the less end, so is the height of the cylinder to the part of the side produced.*

(2) *Find the surface of each of the ungulas, thus formed, by Prob. LXXXIV., and their difference will be the convex surface of the frustum of the cylindric ungula.*

Prob. LXXXVII. To find the volume of a frustum of a cylindric ungula.

$$\textit{Formula.}—V=\frac{a}{b}\Big[\frac{2}{3}(2rb-b^2)^{\frac{3}{2}}-(r-b)\Big\{\frac{1}{2}\pi r^2-$$

$$(r-b)\sqrt{2rb-b^2}-r^2\sin^{-1}\frac{r-b}{r}\Big\}\Big]-\frac{a'}{b'}\Big[\frac{2}{3}(2rb'-b'^2)^{\frac{3}{2}}-$$

$(r-b')\left\{\frac{1}{2}\pi r^2-(r-b')\sqrt{(2rb'-b'^2)}-r^2\sin^{-1}\frac{r-b'}{r}\right\}\Big]$.

Rule.—*Find the volume of the ungula whose base is the the upper base of the frustum and altitude that as found by (1) of the last rule. Also the volume of the ungula whose base is the lower base of the frustum and altitude the sum of the less ungula and altitude of the frustum. Their difference will be the volume of the frustum.*

3. PYRAMID AND CONE.

Prob. LXXXVIII. To find the convex surface of a right cone.

Formula.—$S=C\times\frac{1}{2}h=2\pi r\times\frac{1}{2}\sqrt{a^2+r^2}$, where C is the circumference, h the slant height, r the radius of the base, and a the altitude.

Rule.—*Multiply the circumference of the base by the slant height and take half the product. Or, if the altitude and radius of the base are given, multiply the circumference of the base by the square root of the sum of the squares of the radius and altitude, and take half the product.*

I. What is the convex surface of a right cone whose altitude is 8 inches and the radius of whose base is 6 inches?.

By formula, $S=2\pi r\times\frac{1}{2}\sqrt{a^2+r^2}=2\pi 6\times\frac{1}{2}\sqrt{8^2+6^2}=160\pi=$ 188.495559 sq. in.

II.
1. 6 in.=the radius AD of the base, and
2. 8 in.=the altitude CD.
3. 10 in.$=\sqrt{8^2+6^2}=$the slant height CA.
4. 37.6991118 in.$=2\pi r=12\times$ 3.14159265=the circumference of the base.
4. ∴ 188.495559 sq. in=$\frac{1}{2}(10\times37.6991118)$=the convex surface of the cone.

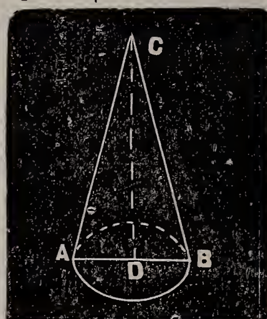

FIG. 40.

III. ∴ The convex surface of the cone is 188.495559 sq. in.

Prob. LXXXIX. To find the convex surface of a pyramid.

Formula.—$S=\frac{1}{2}p\times h$, in which p is the perimeter of the base and h the slant height.

Rule.—*Multiply the perimeter of the base by the slant height and take half the product.*

I. What is the convex surface of a pentagonal pyramid whose slant height is 8 inches and one side of the base 3 inches?

By formula, $S=\frac{1}{2}p\times h=\frac{1}{2}(3+3+3+3+3)\times 8=60$ sq. in.

II. $\begin{cases} \text{1. 8 in.=the slant height.} \\ \text{2. 3 in.=the length of one side of the base.} \\ \text{3. } 5\times 3 \text{ in.=15 in.=the perimeter of the base.} \\ \text{4. } \therefore \frac{1}{2}(15\times 8)=60 \text{ sq. in.=the convex surface of the pyramid.} \end{cases}$

III. ∴ The convex surface of the pyramid is 60 sq. in.

Remark.—If the entire surface of a pyramid or cone is required, to the convex surface add the area of the base.

Formula.—$T=S+A$, where A is the area of the base and S the convex surface.

Prob. XC. To find the volume of a pyramid or a cone.

Formula.—$V=\frac{1}{3}aA=\frac{1}{3}a\times\pi r^2$, where a is the altitude and $A=\pi r^2$ the area of the base.

Rule.—*Multiply the area of the base by the altitude and take one-third of the product.*

I. What is the volume of a cone whose altitude CD is 18 inches and the radius AD of the base 3 inches?

By formula, $V=\frac{1}{3}a\times\pi r^2=\frac{1}{3}\times 18\times\pi 3^2=54\times 3.14159265=$ 169.646 cu. in.

II. $\begin{cases} \text{1. 18 in.=the altitude } CD\text{, and} \\ \text{2. 3 in.=the radius } AD. \\ \text{3. 28.27433385 sq. in.} =\pi r^2=3^2\pi=\text{the area of the base.} \\ \text{4. } \therefore \text{169.6460031 cu. in.}=\frac{1}{3}aA=\frac{1}{3}\times 18\times 3^2\pi=\text{the volume of the cone.} \end{cases}$

III. ∴ The volume of the cone is 169.6460031 cu. in.

Prob. XCI. To find the convex surface of a frustum of a cone.

Formula.—$S=\frac{1}{2}(C+C')h=\frac{1}{2}(2\pi r+2\pi r')h=\pi(r+r')\sqrt{a^2+(r-r')^2}$, in which C is the circumference of the lower base, C' the circumference of the upper base, and $h,=\sqrt{a^2+(r-r')^2}$, the slant height.

Rule.—*Multiply half the sum of the circumferences of the two bases by the slant height.*

1. What is the convex surface of the frustum of a cone whose altitude is 4 feet, radius of the lower base 4 feet, and the radius of the upper base 1 foot?

By formula, $S=\pi(r+r')\sqrt{a^2+(r-r')^2}=\pi(4+1)\sqrt{4^2+(4-1)^2}$ $=25\pi.=78.539816$ sq. ft.

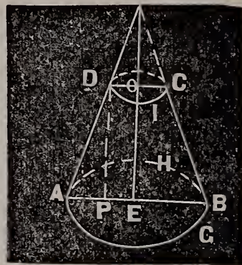

FIG. 41.

II.
1. 4 ft.= the altitude OE,
2. 4 ft.=the radius AE of the lower base, and
3. 1 ft.=the radius DO of the upper base.
4. 3 ft.$=AE-PE(=DO)=r-r'$.
5. 5 ft.$=\sqrt{(DP^2+AP^2)}=\sqrt{a^2+(r-r')^2}=\sqrt{4^2+(4-1)^2}$ $=AD$, the slant height.
6. 8π=the circumference $AGBH$ of the lower base.
7. 2π=the circumference DIC of the upper base.
8. $5\pi=\frac{1}{2}(8\pi+2\pi)$=half the sum of the circumferences.
9. $\therefore$ $5\times5\pi=25\pi=78.539816$ sq. ft.=the convex surface of the frustum.

III. $\therefore$ The convex surface of the frustum is 78.539816 sq. ft.

Remark.—If the entire surface of the frustum is required, to the convex surface add the area of the two bases.

Formula.—$T=S+A+A'=\pi(r+r')\sqrt{a^2+(r-r')^2}+\pi r^2+\pi r'^2$.

Prob. XCII. To find the convex surface of the frustum of a pyramid.

Formula.—$S=\frac{1}{2}(p+p')h$.

Rule.—*Multiply half the sum of the perimeters of the two bases by the slant height.*

I. What is the convex surface of the frustum of a pentagonal pyramid, if each side of the lower base is 5 feet, each side of the upper base 1 foot, and the altitude of the frustum 10 feet?

Before we can apply the formula, we must find the slant height. Produce FO, till $OK=OE$. Divide OK into extreme and mean ratio at H. Draw EH. Then $KO:OH::OH:KH$. $\therefore$ $OH^2=KO\times KH=KO\times(KO-OH)=KO^2-KO\times OH$; whence $OH^2+KO\times OH=KO^2$. Completing the square of this equation, $OH^2+KO\times OH+\frac{1}{4}KO^2=\frac{5}{4}KO^2$, from which $OH(=EH=EK)=\frac{1}{2}KO(\sqrt{5}-1)$. $EF^2=EK^2-KF^2=[\frac{1}{2}KO(\sqrt{5}-1)]^2-[\frac{1}{2}(KO-OH)]^2,=\frac{1}{4}KO^2(\sqrt{5}-1)^2-\left[\frac{1}{2}\left\{KO-\frac{1}{2}KO(\sqrt{5}-1)\right\}\right]^2=\frac{1}{4}KO^2(\sqrt{5}-1)^2-\frac{1}{4}KO^2(3-\sqrt{5})^2=\frac{1}{4}KO^2\Big[(\sqrt{5}-1)^2-$

$\frac{1}{4}(3-\sqrt{5})^2\Big]=\frac{1}{4}KO^2\left[\frac{10-2\sqrt{5}}{4}\right]=\frac{1}{16}KO^2(10-2\sqrt{5})$. But $EF=\frac{1}{2}EA=\frac{1}{2}s$. $\therefore$ $\frac{1}{4}s^2=\frac{1}{16}KO^2(10-2\sqrt{5})$, and $s=\frac{1}{2}KO\sqrt{10-2\sqrt{5}}$.

$\therefore$ $KO=\frac{2s}{\sqrt{10-2\sqrt{5}}}$, where s is a side of the lower base, $=\frac{10}{\sqrt{10-2\sqrt{5}}}$. KO may be considered the radius R of a circumscribed circle of the lower base. In like manner, the radius r of the circumscribed circle of the upper base may be found to be $\frac{2s'}{\sqrt{10-2\sqrt{5}}}$, where s' is a side of the upper base, $=\frac{2}{\sqrt{10-2\sqrt{5}}}$.

OF, the apothem of the lower base, $=\sqrt{(EO^2-EF^2)}=\sqrt{\left[\left(\frac{10}{\sqrt{10-2\sqrt{5}}}\right)^2-\left(\frac{5}{2}\right)^2\right]}=\frac{5}{2}\sqrt{\left(\frac{3+\sqrt{5}}{5-\sqrt{5}}\right)}$. In like manner, $fo=\frac{1}{2}\sqrt{\left(\frac{3+\sqrt{5}}{5-\sqrt{5}}\right)}$. $\therefore$ $IF=OF-OI(=fo)=\frac{5}{2}\sqrt{\left(\frac{3+\sqrt{5}}{5-\sqrt{5}}\right)}-\frac{1}{2}\sqrt{\left(\frac{3+\sqrt{5}}{5-\sqrt{5}}\right)}=2\sqrt{\left(\frac{3+\sqrt{5}}{5-\sqrt{5}}\right)}$. $Ff=\sqrt{(If^2+IF^2)}=\sqrt{\left\{10^2+\left[2\sqrt{\left(\frac{3+\sqrt{5}}{5-\sqrt{5}}\right)}\right]^2\right\}}=\frac{2}{5}\sqrt{650+10\sqrt{5}}=$the slant height.

By formula, $S=\frac{1}{2}(25+5)\frac{2}{5}\sqrt{650+10\sqrt{5}}=6\sqrt{650+10\sqrt{5}}=$ 155.5795 sq. ft.

II.
1. 10 ft.=the altitude oO.
2. 5 ft.$=EA$, one of the equal sides of the lower base.
3. 1 ft.$=ed$, one of the equal sides of the upper base.
4. $\frac{2}{5}\sqrt{650+10\sqrt{5}}=fF$, the slant height.
5. 5×5 ft.=25 ft.=the perimeter of the lower base.
6. 5×1 ft.=5 ft.=the perimeter of the upper base.
7. $\therefore$ $\frac{1}{2}(25+5)\frac{2}{5}\sqrt{650+10\sqrt{5}}$ = 155.5795 sq.ft.=the convex surface.

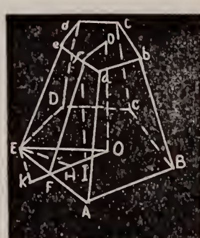

FIG. 42.

III. $\therefore$ The convex surface of the frustum is 155.5795 sq. ft.

Prob. XCIII. To find the volume of a frustum of a pyramid or a cone.

Formula.—(*a*) $V=\frac{1}{3}a(A+\sqrt{AA'}+A')$, in which A is the area of the lower base, A' the area of the upper base and $\sqrt{AA'}$ the area of the mean base. When we have a frustum of a cone, (*b*) $V=\frac{1}{3}a(A+\sqrt{AA'}+A')=\frac{1}{3}a(\pi R^2+\sqrt{(\pi R^2\times\pi r^2)}+\pi r^2)=\frac{1}{3}a(\pi R^2+\pi Rr+\pi r^2)=\frac{1}{3}\pi a(R^2+Rr+r^2)$.

Rule.—(1) *Find the area of the mean base by multiplying the area of the upper and lower bases together and extracting the square root of the product.*

(2) *Add the upper, lower, and mean bases together and multiply the sum by $\frac{1}{3}$ the altitude.*

I. What is the solidity of a frustum of a cone whose altitude is 8 feet, the radius of the lower base 2 feet, and the radius of the upper base 1 foot?

By formula (*b*), $V=\frac{1}{3}\pi a(R^2+Rr+r^2)=\frac{1}{3}\pi 8(4+2+1)=\frac{1}{3}\times 56\pi=58.6433$ cu. ft.

II.
1. 8 ft.=the altitude.
2. 2 ft.=the radius of the lower base.
3. 1 ft.=the radius of the upper base.
4. 4π=the area of the lower base.
5. π=the area of the upper base.
6. $2\pi=\sqrt{4\pi\times\pi}$=the area of the mean base.
7. $4\pi+\pi+2\pi=7\pi$=the sum of the areas of the three bases.
8. $\therefore \frac{1}{3}\times 8\times 7\pi=58.6433$ cu. ft.=the solidity of the frustum.

III. $\therefore$ The solidity of the frustum is 58.6433 cu. ft

4. CONICAL UNGULAS.

1. ***A Conical Ungula*** (Lat. *ungula*, a claw, hoof, from *unguis*, a nail, claw, hoof) is a section or part of a cone cut off by a plane oblique to the base and contained between this plane and the base.

Prob. XCIV. To find the surface of a conical ungula.

$$\textit{Formula.}\text{—}S\int_r^R s\sqrt{dx^2+dy^2}=$$

$$\frac{1}{R-r}\sqrt{a^2+(R-r)^2}\int_r^R sdx=\frac{\sqrt{a^2+(R-r)^2}}{R-r}\int_r^R\left\{2\pi x-2x\cos^{-1}\left[\frac{(2R-t)r-(R+r-t)x}{R-r}\right]\right\}dx,$$

where a is the altitude of the ungula, R the radius of the base, r the radius of the upper base of the frustum from which the ungula is cut, t the distance the cutting plane cuts the base from the opposite extremity of the base, and x the radius of a section parallel to the base and at a distance $h-y$ from the base.

Prob. XCV. To find the volume of a conical ungula.

Formula.— $V=\int_r^R A dy=$

$$\frac{a}{R-r}\int_r^R \left\{ x^2\cos^{-1}\frac{(2R-t)r-(R+r-t)x}{(R-r)x}+\frac{1}{(R-r)^2}\Big[(2R-t)r-(R+r-t)x\Big]\sqrt{\Big[-(2R-t)^2r^2+2r(2R-t)(R+r-t)x-(2R-t)(2r-t)x^2\Big]}\right\} dx,$$

where the letters represent the same value as in the preceding problem and $dy=\left(\frac{a}{R-r}\right)dx$, since $y=\frac{a(x-r)}{R-r}$.

Prob. XCVI. To find the convex surface of a conical ungula, when the cutting plane passes through the opposite extremities of the ends of the frustum.

Formula.—$S=$

$$\frac{\pi}{R-r}\sqrt{a^2+(R-r)^2}\left\{R^2-\tfrac{1}{2}(R+r)\sqrt{Rr}\right\}.$$

This formula is obtained by putting $t=0$, in the formula of Prob. XCIV., and integrating the result. For, in this problem, the cutting plane $AHCK$ passes through the opposite point A, and therefore the distance from A to the cutting plane is 0. $\therefore\ t=0$.

FIG. 43.

Rule.—*Multiply half the sum of the radii of the bases by the square root of their product and subtract the result from the square of the radius of the lower base. Multiply this difference by π times the slant height and divide the result thus obtained by the difference of the radii of the bases.*

Prob. XCVII. To find the volume of a conical ungula, when the cutting plane passes through the opposite extremities of the ends of the frustum.

Formula.— $V=\frac{\pi R^{\frac{3}{2}}a}{3(R-r)}\left(R^{\frac{3}{2}}-r^{\frac{3}{2}}\right)$.

This formula is obtained by putting $t=0$, in the formula of Prob. XCV., and integrating the result.

Rule.—*Multiply the difference of the square roots of the cubes of the radii of the bases by the square root of the cube of the radius of the lower base and this product by $\frac{1}{3}\pi$ times the altitude.*

Divide this last product by the difference of the radii of the two bases and the quotient will be the volume of the ungula.

I. A cup in the form of a frustum of a cone is 7 in. in diameter at the top, 4 in. at the bottom, and 6 in. deep. If, when full of water, it is tipped just so that the raised edge of the bottom is visible; what is the volume of the water poured out?

By formula, $V=\dfrac{\pi R^{\frac{3}{2}}a}{3(R-r)}\left(R^{\frac{3}{2}}-r^{\frac{3}{2}}\right)=\frac{7}{6}\pi(49-8\sqrt{7})=$ 102.016989 cu. in.

Remark.—Fig. 43 inverted represents the form of the cup and $APBQ-C$ the quantity of water poured out, C being the tipped edge of the bottom.

I. A tank is 6 feet in diameter at the top, 8 feet at the bottom, and 12 feet deep. A plane passes from the top on one side to the bottom on the other side: into what segments does it divide the tank?

By formula, $V=\dfrac{\pi R^{\frac{3}{2}}a}{3(a-b)}(R^{\frac{3}{2}}-r^{\frac{3}{2}})=\dfrac{96\pi}{3(4-3)}(8-3\sqrt{3}),=$ $32\pi(8-3\sqrt{3})=281.87$ cu ft.

II.
1. 4 ft.$=AL$, the radius of the lower base.
2. 3 ft.$=DF$, the radius of the upper base, and
3. 12 ft.$=FL$, the altitude. Then
4. $\dfrac{\pi\sqrt{4^3}\times12}{3(4-3)}(\sqrt{4^3}-\sqrt{3^3})=32\pi(8-3\sqrt{3})=281.87$ sq. ft.=the volume.

III. ∴ The volume is 281.87 cu. ft.

Prob. XCVIII. To find the convex surface of a conical ungula, when the cutting plane FCE makes an angle CIB less than the angle DAB, i. e. when AI(=t) is less than DC(=2r).

$$\textit{Formula.}—S=\frac{1}{R-r}\sqrt{a^2+(R-r)^2}\left\{R^2\cos^{-1}\left(\frac{-R+t}{R}\right)\right.$$
$$\left.-\frac{r}{2r-t}(R-r)\sqrt{(2R-t)t}-\frac{r^2(R+r-t)}{2r-t}\sqrt{\frac{2R-t}{2r-t}}R^2\cos^{-1}\left(\frac{r-t}{r}\right)\right.$$

This formula is obtained by integrating the formula of Prob. XCIV, recollecting that the co- efficient of x^2 is negative.

Prob. XCIX. To find the volume of a conical ungula, when the cutting plane FCE makes an angle CIB less than the angle DAB. i. e., when AI (=t) is less than CD (=2r).

FIG. 44.

Formula.— $V=\frac{a}{R-r}\left\{\frac{1}{3}R^3\cos^{-1}\left(\frac{-R+t}{r}\right)\right.$
$-\frac{2}{3}\left[\frac{Rr(R-r)}{t-2r}\right]\sqrt{(2R-t)t}+\frac{(R+r-t)(R-r)}{(2R-t)(t-2r)}\times$
$\left[(2R-t)t\right]^{\frac{3}{2}}+\frac{1}{3}r^3\left(\frac{2(R+r)t-4Rr-t^2}{(t-2r)^2}\right)$
$\left.\sqrt{\frac{2R-t}{t-2r}}\cos^{-1}\left(\frac{r-t}{r}\right)\right\}.$

This formula is obtained by integrating the formula of Prob. XCV, recollecting that the coefficient of x^2 is negative.

Prob. C. To find the convex surface of a conical ungula, when the cutting plane FCE is parallel to the side AD, i. e., when AI(=t) is equal to DC(=2r).

$$\textit{Formula.}—S=\frac{1}{R-r}\sqrt{a^2+(R-r)^2}\left\{R^2\cos^{-1}\left(\frac{-R+2r}{R}\right)\right.$$

$$\left.+2(R-2r)\sqrt{(R-r)r}-\frac{8}{3}(R-r)\sqrt{(R-r)r}\right\}$$

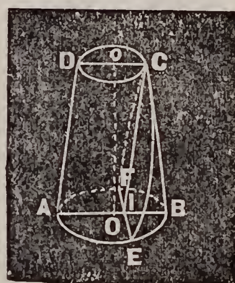

FIG 45.

This formula is obtained by putting $t=2r$, in the formula of Prob. XCIV., and integrating the resulting equation.

Pob. CI. To find the volume of a conical ungula, when the cutting plane FCE is parallel to the side DA, i. e., when AI(=t) is equal to CD (=2r).

$$\textit{Formula.}— V=\frac{1}{3}a\left\{\frac{R}{R-r}\left[R^2\cos^{-1}\left(\frac{-R+2r}{R}\right)+\right.\right.$$

$$\left.\left.2(R-2r)\sqrt{(R-r)r}\right]-\frac{16}{3}r\sqrt{(R-r)r}\right\}.$$

This formula is obtained by putting $t=2r$, in the formula of Prob. XCV., and integrating the resulting equation.

Prob. CII. To find the convex surface of a concial ungula, when the cutting plane FCE makes an angle CIB greater than the angle DAB, i. e., when AI (=t) is greater than DC (=2r).

$$\textit{Formula.}— S=\frac{1}{R-r}\sqrt{a^2+(R-r)^2}\left\{R^2\cos^{-1}\left(\frac{-R+t}{R}\right)\right.$$

$$-\frac{r}{t-2r}(R-r)\sqrt{(2R-t)t}-\frac{r^2(R+r-t)}{t-2r}\sqrt{\frac{2R-t}{t-2r}}\log\left[\left\{t-r+\right.\right.$$

$$\left.\left.\left.(t-2r)\sqrt{\frac{t}{t-2r}}\right\}\div r\right]\right\}.$$

This formula is obtained by integrating the formula of Prob. XCIV., remembering that the coefficient of x^2, which occurs in process of integrating, is positive.

Prob. CIII. To find the volume of a conical ungula, when the cutting plane FCE makes an angle CIB greater than the angle DAB, i. e., when AI(=t) is greater than DC(=2r).

Formula.— $V=\frac{a}{R-r}\Big\{\frac{1}{3}R^3\cos^{-1}\left(\frac{-R+t}{R}\right)-$

$\frac{2}{3}\left[\frac{Rr(R-r)}{t-2r}\right]\sqrt{(2R-t)t}+$

$\frac{(R+r-t)(R-r)}{(2R-t)(t-2r)}[(2R-t)t]^{\frac{3}{2}}+$

$\frac{1}{3}r^3\left(\frac{2(R+r)t-4Rr-t^2}{(t-2r)^2}\right)\sqrt{\frac{2R-t}{t-2r}}\times$

$\log\left[\left(t-r+(t-2r)\sqrt{\frac{t}{t-2r}}\right)\div r\right]$.

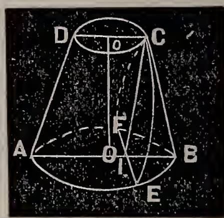

FIG. 46.

This formula is obtained by integrating the formula of Prob. XCV., regarding the coefficient of x^2 positive.

XII. THE SPHERE.

Prob. CIV. To find the convex surface of a sphere.

Formula.—$S=2\times2\pi y\sqrt{dy^2+dx^2}=4\pi R^2=\pi D^2$, where D is the diameter.

Rule.—*Multiply the square of the diameter by 3.141592.*

I. What is the surface of a sphere whose radius is 5 inches?

By formula, $S=4\pi R^2=4\pi\times25=314.1592$ sq. in.

II.
1. 5 in.=the radius.
2. 25 sq. in.= the square of the radius.
3. ∴ $4\pi\times25$ sq. in.=314.1592 sq. in.=the surface of the sphere.

III. ∴ 314.1592 sq. in.=the surface of the sphere.

NOTE.—Since πR^2 is the area of a circle whose radius is R, the area ($4\pi R^2$) of a sphere is equal to four great circles of the sphere. The surface of a sphere is also equal to the convex surface of its circumscribing cylinder.

Prob. CV. To find the volume of a sphere, or a globe.

Formula.— $V=2\pi y^2dx=\frac{4}{3}\pi R^3=\frac{4}{3}\pi(\frac{1}{2}D)^3=\frac{1}{6}\pi D^3$.

Rule.—*Multiply the cube of the radius by $\frac{4}{3}\pi$ (=4.188782); or multiply the cube of the diameter by $\frac{1}{6}\pi$(=.5235987).*

I. What is the volume of a sphere whose diameter is 4 feet?

By formula, $V=\frac{4}{3}\pi R^3=\frac{4}{3}\pi 2^3=33.510256$ cu. ft.

II. 1. 2 ft.=the radius.
2. 8 cu. ft.$=2^3$=the cube of the radius.
3. ∴ 4.188782×8 cu. ft.=33.510256 cu. ft.=the volume of the sphere.

III. ∴ 33.510256 cu. ft.=the volume of the sphere.

Prob. CVI. To find the area of a zone.

A Zone is the curved surface of a sphere included between two parallel planes or cut off by one plane.

Formula.—$S=2\pi Ra$, in which a is the altitude of the segment of which the zone is the curved surface.

Rule.—*Multiply the circumference of a great circle of the sphere by the altitude of the segment.*

I. What is the area of a zone whose altitude is 2 feet, on a sphere whose radius is 6 feet?

By formula, $S=2\pi Ra=2\pi 6\times 2=24\pi=75.39822$ sq. ft.

II. 1. 6 ft.=the radius of the sphere.
2. 2 ft.=the altitude.
3. $12\pi=37.69911$ ft.=the circumference of a great circle of the sphere.
4. ∴ 2×37.69911=75.39822 sq. ft.=the area of the zone.

III. ∴ The area of the zone is 75.39822 sq. ft.

NOTE.—This rule is applicable whether the zone is the curved surface of the frustum of a sphere or the curved surface of a segment of a sphere.

Prob. CVII. To find the volume of the segment of a sphere.

Formula.—$V=\frac{1}{6}\pi a(3r_1^2+a^2)$ where r_1 is the radius of the base of the segment.

Rule.—*To three times the square of the radius of the base, add the square of the altitude and multiply the sum by $\frac{1}{6}\pi=.5235987$ times the altitude.*

I. What is the volume of a segment whose altitude is 2 inches and the radius of the base 8 inches?

By formula, $V=\frac{1}{6}\pi a(3r_1^2+a^2)=\frac{1}{6}\pi\times 2(3\times 64+4)=205.2406$ cu. in.

II. 1. 8 in.=the radius of the base.
2. 2 in.=the altitude of the segment.
3. 192 sq. in $=3\times 8^2$=three times the square of the radius.
4. 4 sq. in.=the square of the altitude.
5. 196 sq. in.=192 sq. in.+4 sq. in.=three times the square of the radius plus the square of the altitude.
6. $\frac{1}{6}\pi\times 2\times 196=205.2406$ cu. in.=the volume of the segment.

III. ∴ 205.2406 cu. in =the volume of the segment.

NOTE.—From the formula $V=\frac{1}{6}\pi a(3r_1^2+a^2)$, we have $V=\frac{1}{2}\pi ar_1^2+\frac{1}{6}\pi a^3$ But $\frac{1}{2}\pi ar_1^2$ is the volume of a cylinder whose radius is r_1, and altitude $\frac{1}{2}a$, and $\frac{1}{6}\pi a^3$ is the volume of a sphere whose diameter is a ∴ The volume of a segment of a sphere is equal to a cylinder whose base is the base of the segment and altitune half the altitude of the segment, plus a sphere whose diameter is the altitude of the segment.

Prob. CVIII. To find the volume of a frustum of a sphere ,or the portion included between two parallel planes.

Formula.— $V=\frac{1}{6}\pi a[3(r_1^2+r_2^2)+a^2]=\frac{1}{2}a(\pi r_1^2+\pi r_2^2)+\frac{1}{6}\pi a^3$ *, in which r_1 is the radius of the lower base, r_2 the radius of the upper base.

Rule.—*To three times the sum of the squared radii of the two ends, add the square of the altitude; multiply this sum by .5235987 times the altitude.*

I. What is the volume of the frustum of a sphere, the radius of whose upper base is 2 feet and lower base 3 feet and altitude $\frac{1}{2}$ foot?

By formula, $V=\frac{1}{6}\pi a[3(r_1^2+r_2^2)+a^2]=\frac{1}{6}\pi\times\frac{1}{2}[3(9+4)+\frac{1}{4}]=$ 8.03839 cu. ft.

II.
1. 3 ft.=the radius of the lower base.
2. 2 ft.=the radius of the upper base.
3. 39 sq. ft.$=3(3^2+2^2)$=three times the sum of the squares of the radii of the two bases.
4. $\frac{1}{4}$ sq. ft.=the square of the altitude.
5. ∴ $\frac{1}{6}\pi\times\frac{1}{2}\times39\frac{1}{4}=8.03839$ cu. ft.=the volume of the frustum.

III. ∴ 8.03839 cu. ft.=the volume of the frustum.

Prob. CIX. To find the volume of spherical sector.

A Spherical Sector is the volume generated by any sector of a semi-circle which is revolved about its diameter.

Formula.— $V=\frac{2}{3}\pi aR^2$, where a is the altitude of the zone of the sector.

Rule.—*Multiply its zone by one-third the radius.*

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* NOTE.—$\frac{1}{2}a(\pi r_1^2+\pi r_2^2)$=the volume of two cylinders whose bases are the upper and lower bases of the segment and whose altitude is half the altitude of the segment. $\frac{1}{6}\pi a^3$ is the volume of a sphere whose diameter is the altitude of the segment. Hence the volume of a segment of a sphere of two bases is equivalent to the volume of two cylinders whose bases are the upper and lower bases respectively of the segment and whose common altitude is the altitude of the segment, plus the volume of a sphere whose diameter is the altitude of the segment.

For a demonstration of this and the preceding formula, see *Wentworth's Plane and Solid Geometry, Bk. IX., Prob. XXXII.*

I. What is the volume of a spherical sector the altitude of whose zone is 2 meters and the radius of the sphere 6 meters?

By formula, $V=\frac{2}{3}\pi a R^2=\frac{2}{3}\pi\times2\times6^2=$ 150.7964m³.

FIG. 47.

II.
1. 2m.=the altitude BD of the zone generated by the arc EF when the semicircle is revolved about AB.
2. 6m.=the radius EC of the sphere.
3. $2\pi 6$m.=37.699104 m =the circumference of a great circle of the sphere.
4. $2\pi 6\times2=75.398208$ m².=the area of the zone generated by EF, by Prob. CVI.
5. $\therefore \frac{1}{3}\times6\times75.398208=150.796416$m³=the volume of the spherical sector.

III. ∴ The volume of the spherical sector is 150.796416 m³.

I. Find the diameter of a sphere of which a sector contains 7853.98 cu. ft.. when the altitude of its zone is 6 feet.

By formula, $V=\frac{2}{3}\pi a r^2=\frac{2}{3}\pi\times6\times r^2$. $\therefore \frac{2}{3}\pi\times6\times r^2=$ 7853.98 cu. ft., or $4r^2=2500$ sq. ft., whence $2r=50$ feet, the diameter of the sphere.

II.
1. 6 ft.=the altitude of the zone.
2. $\therefore \frac{2}{3}\pi\times6\times r^2$=the volume of the sector. But
3. 7853.98 cu. ft.=the volume.
4. $\therefore \frac{2}{3}\pi\times6\times r^2=7853.98$ cu. ft.
5. $r^2=625$ sq. ft, by dividing by 4π.
6. $\therefore 2r=50$ ft., the diameter of the sphere.

III. ∴ The diameter of the sphere is 50 feet.

Prob. CX. To find the area of a lune.

A Lune is that portion of a sphere comprised between two great semi-circles.

Formula.—$S=4\pi R^2\left(\frac{A}{360^\circ}\right)=4\pi R^2 u$, where u is the quotient of the angle of the lune divided by 360°.

Rule.—*Multiply the surface of the sphere by the quotient of the angle of the lune divided by 360°*

I. Given the radius of a sphere 10 inches; find the area of a lune whose angle is 30°.

By formula, $S=4\pi R^2 u=4\times\pi\times10^2\times(30^\circ\div360^\circ)=$ $\frac{1}{3}\pi10^2=104.7197$ sq. in.

II.
1. 10 in.=the radius of the sphere.
2. 30°=the angle of the lune.
3. $\frac{1}{12}=30^\circ\div 360^\circ$=the quotient of the angle of the lune divided by 360°.
4. $4\pi 10^2=400\pi=1256.6368$ sq. in.=the surface of the sphere.
5. $\therefore \frac{1}{12}\times 1256.6368$ sq. in.$=104.7198$ sq. in.=the area of the lune.

III. $\therefore$ The area of the lune is 104.7198 sq. in.

Wentworth's New Plane and Solid Geometry, p. 371, Ex. 585.

Prob. CXI. To find the volume of a spherical ungula.

A Spherical Ungula is a portion of a sphere bunded by a *lune* and two great semi-circles.

Formula.— $V=\frac{4}{3}\pi R^3 u$, where u is the same as in the last problem.

Rule.—*Multiply the area of the lune by one-third the radius; or, multiply the volume of the sphere by the quotient of the angle of the lune divided by 360°.*

I. What is the volume of a spherical ungula the angle of whose lune is 20°, if the radius of the sphere is 3 feet?

By formula, $V=\frac{4}{3}\pi R^3 u=\frac{4}{3}\pi\times 3^3\times(20^\circ\div 360^\circ)=6.283184$ cu. ft.

II.
1. 3 ft.=the radius of the sphere.
2. $4\pi 3^2\times(20^\circ\div 360^\circ)=6.283184$ sq. ft.=the area of the lune, by Prob CX
3. $\therefore \frac{1}{3}\times 3\times 6.283184=6.283184$ cu. ft.=the volume of the ungula.

III. $\therefore$ 6.283184 cu. ft. is the volume of the ungula.

Prob. CXII. To find the area of a spherical triangle.

Formula.—$S=2\pi R^2\times(A+B+C-180^\circ)\div 360^\circ$, in which A, B, and C are the angles of the spherical triangle.

Rule.—*Multiply the area of the hemisphere in which the triangle is situated by the quotient of the spherical excess (the excess of the sum of the spherical angles over 180°) divided by 360°.*

I. What is the area of a spherical triangle on a sphere whose diameter is 12, the angles of the triangle being 82°, 98°, and 100°?

By formula, $S=2\pi R^2\times(A+B+C-180^\circ)\div 360^\circ=2\pi 6^2\times(82^\circ+98^\circ+100^\circ-180^\circ)\div 360^\circ=2\pi 6^2\times\frac{5}{18}=62.83184$=area.

II.
1. 6=the radius of the sphere.
2. $2\pi 6^2=72\pi$=the area of the hemisphere.
3. $(82°+98°+100°-180°)=100°$=the spherical excess.
4. $100°\div360°=\frac{5}{18}$=the quotient of the spherical excess divided by 360°.
5. $\therefore \frac{5}{18}\times72\pi=62.83184$=the area of the spherical triangle.

III. $\therefore$ The area of the spherical triangle is 62.83184.

(*Olney's Geometry and Trigonometry, Un. Ed., p. 238, Ex. 8.*)

Prob. CXIII. To find the volume of a spherical pyramid.

A Spherical Pyramid is the portion of a sphere bounded by a spherical polygon and the planes of its sides.

Formula.— $V=\frac{2}{3}\pi R^3\times(E\div360°)$, where E is the spherical excess.

Rule.—*Multiply the area of the base by one-third of the radius of the sphere*

I. The angles of a triangle, on a sphere whose radius is 9 feet, are 100°, 115°, and 120°; find the area of the triangle and the volume of the corresponding spherical pyramid.

By formula, $V=\frac{2}{3}\pi R^3\times(E\div360°)=\frac{2}{3}\pi R^3\times(A+B+C-180°)\div360°=\frac{2}{3}\pi 9^3\times(100°+115°+120°-180°)\div360°=\frac{31}{108}\pi 9^3=657.377126$ cu. ft.

II.
1. 9 ft.=the radius of the sphere.
2. $2\pi9^2$=the area of the hemisphere in which the pyramid is situated.
3 $(100°+115°+120°-180°)=155°$=the sperical excess.
4. $\frac{31}{72}=155°\div360°$=the quotient of the spherical excess divided by 360°.
5. $\therefore \frac{31}{72}\times2\pi9^2=\frac{31}{36}\times\pi 9^2$=the area of the base of the pyramid.
6. $\therefore \frac{1}{3}\times9\times\frac{31}{36}\times2\pi9^2=657.377126$ cu. ft.=the volume of the pyramid.

III. $\therefore$ The volume of the spherical pyramid is 657.377126 cu. ft.

(*Van Amringe's Davies' Geometry and Trigonometry, p. 278, Ex. 15.*

I. Find the area of a spherical hexagon whose angles are 96°, 110°, 128°, 136°, 140°, and 150°, if the circumference of a great circle of the sphere is 10 inches.

Formula.—$S=2\pi R^2\frac{[T-(n-2)180°]}{360°}$, where T is the sum of the angles of the polygon and n the number of sides.

By formula, $S=2\pi R^2\times\frac{[T-(n-2)180^\circ]}{360^\circ}=2\pi\times\left(\frac{10}{2\pi}\right)^2\times$ $(96^\circ+110^\circ+128^\circ+136^\circ+140^\circ+150^\circ-(6-2)\times180^\circ)\div$ $360^\circ=\frac{50}{\pi}\times(760^\circ-720^\circ)\div360^\circ=\frac{1}{9}\frac{50}{\pi}=1.7684$ sq. in.

II. 1. $5\div\pi$=the radius of the sphere, since $2\pi R=10$ in..
2. $760^\circ=96^\circ+110^\circ+128^\circ+136^\circ+140^\circ+150^\circ$=the sum of the angles of the polygon.
3. $760^\circ-(6-2)\times180^\circ=40^\circ$=the spherical excess.
4. $\frac{1}{9}=40^\circ\div360^\circ$=the quotient of the spherical excess divided by 360°.
5. $2\pi\left(\frac{5}{\pi}\right)^2$=the area of the hemisphere on which the polygon is situated.
6. $\therefore\ \frac{1}{9}\times2\pi\left(\frac{5}{\pi}\right)^2=\frac{1}{9}\times50\div\pi=1.7684$ sq. in.

III. $\therefore$ The area of the polygon is 1.7684 sq. in.

Wentworth's Geometry, Revised Ed., p. 374, Ex. 596.

XIII. SPHEROID.

1. ***A Spheroid*** is a solid formed by revolving an ellipse about one of its diameters as an axis of revolution.

1. THE PROLATE SPHEROID.

1. ***The Prolate Spheroid*** is the spheroid formed by revolving an ellipse about its transverse diameter as an axis of revolution.

Prob. CXIV. To find the surface of a prolate spheroid

Formulae.—$(a)\ S=2\int2\pi y\,ds=2\int2\pi y\sqrt{1+\frac{dy^2}{dx^2}}\,dx=$ $4\pi y\left(\frac{a^4y^2+b^4x^2}{a^4y^2}\right)^{\frac{1}{2}}dx,=\frac{4\pi}{a^2}[a^2(a^2b^2-b^2x^2)+b^4x^2]^{\frac{1}{2}}dx=$ $4\pi\frac{b}{a}(a^2-e^2x^2)^{\frac{1}{2}}dx=2\pi b^2+2\frac{\pi ab}{e}\sin^{-1}e,=2\pi b(b+\frac{a}{e}\sin^{-1}e)$, where $e=\frac{\sqrt{a^2-b^2}}{a}$=the eccentricity of the ellipse which generates the surface.

$(b)\quad S=4\pi ab(1-\frac{e^2}{2.3}-\frac{e^4}{2.4.5}-\frac{3e^6}{2.4.6.7}-\frac{3.5e^8}{2.4.6.8.9}-\&c.)$

Rule.—*Multiply the circumference of a circle whose radius is the semi-conjugate diameter by the semi-conjugate diameter increased by the product of the arc whose sine is the eccentricity into the quotient of the semi-transverse diameter divided by the eccentricity.*

I. Find the surface of a prolate spheroid whose transverse diameter is 10 feet and conjugate diameter 8 feet.

By formula (a); $S=2\pi b(b+\frac{a}{e}\sin^{-1}e)=2\pi 4(4+\frac{5}{e}\sin^{-1}e)=$

$2\pi 4\left[4+\left(5\div\frac{\sqrt{5^2-4^2}}{5}\right)\sin^{-1}\frac{\sqrt{5^2-4^2}}{5}\right]=2\pi 4[4+$

$(5\div\frac{3}{5})\sin^{-1}\frac{3}{5}]=\frac{2}{3}\pi[48+100\sin^{-1}\frac{3}{5}]=\frac{2}{3}\pi[48+100\times\frac{53093}{259200}\pi]=$

$\frac{2}{3}\pi[48+100\times.6435053]=235.3064$ sq. ft.

II.
1. $25.1327412=2\pi 4=$the circumference of a circle whose radius is the semi-conjugate diameter of the ellipse.
2. $\frac{3}{5}=\frac{\sqrt{5^2-4^2}}{5}=$the eccentricity.
3. $\frac{25}{3}$ ft.$=5$ ft$\div\frac{3}{5}=$the quotient of the semi-transverse diameter divided by the eccentricity.
4. $.6435053=$the arc (to the radius 1) whose sine is $\frac{3}{5}$, or the eccentricity.
5. 5.3625442 ft.$=\frac{25}{3}$ft. $\times.6435053=\frac{25}{3}$ ft. $\times$ the arc whose sine is $\frac{3}{5}$
6. 9.3625442 ft.$=4$ ft.$+5.3625442$ ft.$=$semi-conjugate diameter increased by said product.
7. $\therefore$ 235.3064 sq. ft.$=9.3625442\times25.1327412=$the surface of the prolate spheroid.

III. $\therefore$ The surface of the prolate spheroid is 235.3064 sq. ft.

Prob. CXV. To find the volume of a prolate spheroid.

Formula.—$V=\int\pi y^2dx=\pi\frac{b^2}{a^2}\int_{-a}^{a}(a^2-x^2)dx=$

$\pi\frac{b^2}{a^2}\left[a^2x-\frac{1}{3}x^3\right]_{-a}^{a}=\frac{4}{3}\pi b^2a$, in which b is the semi-conjugate diameter, and a the semi-transverse diameter.

Rule.—*Multiply the square of the semi-conjugate diameter by the semi-transverse diameter and this product by $\frac{4}{3}\pi$.*

I. What is the volume of a prolate spheroid, whose semi-transverse diameter is 50 inches, and semi-conjugate diameter 30 inches.

By formula, $V=\frac{4}{3}\pi b^2a=\frac{4}{3}\pi 30^2\times50=188495.559$ cu. in.

II.
1. 30 in.=the semi-conjugate diameter,
2. 50 in.=the semi-transverse diameter.
3. 900 sq. in=the square of the semi-conjugate diameter.
4. 45000 cu. in.=50×900=the square of the semi-conjugate diameter by the semi-transverse diameter.
5. $\therefore \frac{4}{3}\pi 45000=\frac{4}{3}\times3.14159265\times45000$ cu. in.= 188495.559 cu. in.=the volume of the prolate spheroid.

III. $\therefore$ The volume of the prolate spheroid is 188495.559 cu. in.

2. THE OBLATE SPHEROID.

1. ***An Oblate Spheroid*** is the spheroid formed by revolving an ellipse about its conjugate diameter as an axis of revolution.

Prob. CXVI. To find the surfae of an oblate spheroid.

Formulae.—(*a*) $S=\int 2\pi x ds=2\int_{-a}^{a} 2\pi x\sqrt{1+\frac{dx^2}{dy^2}}\,dy=$
$2\pi a^2\left(1+\frac{1-e^2}{2e}\log\left\{\frac{1+e}{1-e}\right\}\right)$.

(*b*) $S=4\pi ab(1+\frac{e^2}{2.3}-\frac{e^4}{2.4.5}+\frac{3e^6}{2.4.6.7}-\frac{3.5e^8}{2.4.6.8.9}+\&c.)$

Prob. CXVII. To find the volume of an oblate spheroid.

Formula.— $V=\int\pi x^2 dy=2\int_0^b \pi\frac{a^2}{b^2}(b^2-y^2)dy=\frac{4}{3}\pi a^2 b$.

Rule.—*Multiply the square of the semi-transverse diameter by the semi.conjugae diameter and this product by* $\frac{4}{3}\pi$.

I. What is the volume of an oblate spheroid, whose transverse diameter is 100 and conjugate diameter 60?

By formula, $V=\frac{4}{3}\pi a^2b=\frac{4}{3}\pi50^2\times30=314159.265$.

II.
1. $30=\frac{1}{2}$ of 60= the semi-conjugate diameter.
2. $50=\frac{1}{2}$ of 100=the semi-transverse diameter.
3. $2500=50^2$=the square of the semi-transverse diameter.
4. $75000=30\times2500$=the square of the semi-transverse diameter multiplied by the semi-conjugate diameter.
5. $\therefore \frac{4}{3}\pi\times75000=314159.265$= the volume of the oblate spheroid.

III. $\therefore$ The volume of the oblate spheroid is 314159.265.

NOTE.—Since the volume of a prolate spheroid is $\frac{4}{3}\pi b^2a$. We may write $\frac{4}{3}\pi b^2a=\frac{2}{3}(\pi b^2\times2a)$. But $\pi b^2\times2a$ is the volume of a cylinder the radius of whose base is b and altitude $2a$. $\therefore$ The volume of a prolate spheroid is $\frac{2}{3}$ of the circumscribed cylinder. In like manner, it may be shown that the volume of an oblate spheroid is $\frac{2}{3}$ of its circumscribed cylinder.

The following is a general rule for finding the volume of a spheroid; *Multiply the square of the revolving axis by the fixed axis and this product by* $\frac{1}{6}\pi$.

Prob. CXVIII. To find the volume of the middle frustum of a prolate spheroid, its length, the middle diameter, and that of either of the ends being given.

CASE I.

When the ends are circular, or parallel to the revolving axis.

Formula— $V=\frac{1}{12}\pi(2D^2+d^2)l$, where D is the middle diameter CD, d the diameter HI of an end, and l the length of the frustum.

Rule.— *To twice the square of the middle diameter add the square of the diameter of either end and this sum multiplied by the length of the frustum, and the product again by $\frac{1}{12}\pi$, will give the solidity.*

I. What is the volume of the middle frustum $HIGF$ of a prolate spheroid, if the middle diameter CD is 50 inches, and that of either of the ends HI or FG is 40 inches, and its length OK 18 inches?

By formula, $V=\frac{1}{12}\pi(2D^2+d^2)l=\frac{1}{12}\pi(2\times50^2+40^2)18=$ 31101.767265 cu. in.

II.
1. 50 in.=the middle diameter CD.
2. 40 in.=the diameter of either end as HI.
3. 18 in.=the length OK of the frustum.
4. 5000 sq. in.$=2\times50^2=$twice the square of the middle diameter.
5. 1600 sq. in.$=40^2=$the sqaure of the diameter of either end.
6. 5000 sq. in.+1600 sq. in.=6600 sq. in.
7. $18\times6600=118800$ cu. in.
8. $\therefore\ \frac{1}{12}\pi\times118800$ cu. in.=31101.767265 cu. in.=the volume.

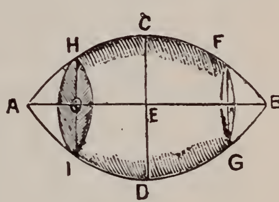


FIG. 48.

III. $\therefore$ The volume of the frustum is 31101.767265 cu. in.

CASE II.

When the ends are elliptical, or perpendicular to the revolving axis.

Formula.— $V=\frac{1}{12}\pi(2Dd+D'd')l$, where D and d are the transverse and conjugate diameters of the middle section and D' and d' the transverse and conjugate diameter of the ends and l the distance between the ends.

Rule.—(1) *Multiply twice the transverse diameter of the middle section by its conjugate diameter, and to this product add*

the product of the transverse and conjugate diameter of either of the ends.

(2) *Multiply the sum, thus found, by the distance of the ends, or the height of the frustum, and the product again by* $\frac{1}{12}\pi$ *and the result will be the volume.*

I. What is the volume of the middle frustum of an oblate spheroid, the diameter of the middle section being 100 inches and 60 inches; those of the end 60 inches and 36 inches; and the length 80 inches?

By formula, $V=\frac{1}{12}\pi(2Dd+D'd')\,l=\frac{1}{12}\pi(2\times100\times60+60\times36)\,80=296566.44616$ cu. in.

II.
1. 100 in.=the transverse diameter FC of the middle section.
2. 60 in.=the conjugate diameter ms of the middle section.
3. 12000 sq. in.$=2\times100\times60$=twice the product of the diameters of the middle section.
4. 60 in.=the transverse diameter AB of the end.
5. 36 in.=the conjugate diameter $2(nc)$ of the end.
6. 2160 sq. in.=the product of the diameters of the end.
7. 14160 sq. in.=12000 sq. in.+2160 sq. in.
8. $80\times14160=1132800$ cu. in. = the product of said sum by the height of the frustum.
9. $\therefore$ $\frac{1}{12}\pi\times1132800$ cu. in.=296566.44616 cu. in.=the volume of the frustum.

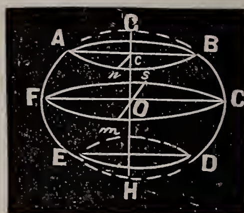


FIG. 49.

III. $\therefore$ The volume of the frustum is 296566.44616 cu. in.

Prob. CXIX. To find the volume of a segment of a prolate spheroid

CASE I.

When the base is parallel to the revolving axis.

Formula.— $V=\frac{1}{6}\pi h^2\left(\frac{d}{D}\right)^2(3D-2h)$, where h is the height of the segment, d the revolving axis, and D the fixed axis.

Rule.—(1) *Divide the square of the revolving axis by the square of the fixed axis, and multiply the quotient by the difference between three times the fixed axis and twice the height of the segment.*

(2) *Multiply the product, thus found, by the square of the height of the segment, and this product by* $\frac{1}{6}\pi$, *and the result will be the volume of the segment.*

I. What is the volume of a segment of a prolate spheroid of which the fixed axis is 10 feet and the revolving axis 6 feet and the height of the segment 1 foot?

By formula, $V=\frac{1}{6}\pi h^2\left(\frac{d}{D}\right)^2(3D-2h)=$
$\frac{1}{6}\pi\times6^2\left(\frac{6}{10}\right)^2(3\times10-2\times1)=$
5.277875652 cu. ft.

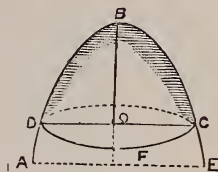


FIG. 50.

II.
1. 10 ft.=the transverse diameter $2BF$.
2. 6 ft.=the conjugate diameter AE.
3. $\frac{9}{25}=\frac{6^2}{10^2}$ =the square of the conjugate diameter divided by the square of the transverse diameter.
4. 28 ft.$=3\times10$ ft.-2×1 ft.= the difference between three times the transverse diameter and twice the height of the segment.
5. $\frac{9}{25}\times28$ ft.$=10\frac{2}{25}$.ft.=the product of said quotient by said difference.
6. $10\frac{2}{25}\times1^2=10\frac{2}{25}$ cu. ft.
7. $\therefore$ $\frac{1}{6}\pi\times10\frac{2}{25}$ cu. ft.=5.277865652 cu. ft.=the volume.

III. $\therefore$ The volume of the segment is 5.277875652 cu. ft.

CASE II.

When the base is perpendicular to the revolving axis.

Formula.— $V=\frac{1}{6}\pi h^2\left(\frac{D}{d}\right)(3d-2h)$, where d is the revolving axis, D the fixed axis, and h the height of the segment.

Rule.—(1) *Divide the fixed axis by the revolving axis, and multiply the quotient by the difference between three times the revolving axis and twice the height of the segment,*

(2). *Multiply the product, thus found, by the square of the height of the segment, and this product again by* $\frac{1}{6}\pi$.

I. Required the volume of the segment of a prolate spheroid, its height being 6 inches, and the axes 50 and 30 inches respectively.

By formula, $V=\frac{1}{6}\pi h^2\left(\frac{d}{D}\right)(3d-2h)=\frac{1}{6}\pi\times6^2\left(\frac{30}{50}\right)\times$

$(3\times30-2\times6)=2450.442267$ cu. in.

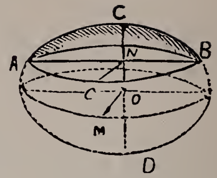


FIG. 51.

II.
1. 50 in.=the transverse diameter, or axis.
2. 30 in.=the conjugate diameter $2MO$.
3. $\frac{5}{3}=50\div30$=the quotient of the trans verse diameter divided by thc conjugate diameter.
4. 78 in.$=3\times30$ in.-2×6in.=the difference between three times the conjugate or revolving axis, and twice the height of the segment.
5. 130 in.$=\frac{5}{3}\times78$ in.=the product of said quotient by said difference.
6. 4680 cu. in$=130\times6^2$=the square of the height of the segment by said product.
7. $\therefore\ \frac{1}{6}\pi\times4680$ cu. in.=2450.442269 cu. in.=the volume of segment.

III. $\therefore$ The volume of the segment is 2450.442269 cu. in.

XIV. CONOIDS.

1. ***A Conoid*** is a solid formed by the revolution of a conic section about its axis.

I. THE PARABOLIC CONOID.

1. ***A Parabolic Conoid*** is the solid formed by revolving a parabola about its axis of abscissa.

Prob. CXX. To find the surface of a parabolic conoid, or paraboloid.

Formulae.—(*a*) $S=\int 2\pi y ds=\int 2\pi y\sqrt{1+\frac{dx^2}{dy^2}}dy=$

$\frac{2\pi y}{p}(p^2+y^2)^{\frac{1}{2}}dy,=\frac{2\pi}{3p}(p^2+y^2)^{\frac{3}{2}}+C=\frac{2\pi}{3}\{(p^2+y^2)^{\frac{3}{2}}-p^3\}$, where $2p$ is the latus rectum of the parabola and y the radius of the base of the conoid, or the ordinate of the parabola.

(*b*) $S=\frac{8}{3}\pi\sqrt{p}\{(p+x)^{\frac{3}{2}}-p^{\frac{3}{2}}\}$, where $2p$ is the same as above and x the altitude of the conoid, or the axis of abscissa of the parabola.

Rule.—*To the square of half the latus rectum, or principal parameter, add the square of the radius of the base of the conoid and extract the square root of the cube of the sum; from this result, subtract the cube of half the latus rectum and multiply the*

difference by 2π, and divide the product by one and one-half times the latus rectum.

I. Determine the convex surface of a paraboloid whose axis is 20, and the diameter of whose base is 60.

From the equation of the parabola, $y^2=2px$, we have $30^2=2p\times20$; whence $2p=45$.

$\therefore$ By formula (a), $S=\frac{2\pi}{3p}\{(p^2+y^2)^{\frac{3}{2}}-p^3\}$

$=\frac{4\pi}{3\times45}\left\{\left[\left(\frac{45}{2}\right)^2+30^2\right]^{\frac{3}{2}}-\left(\frac{45}{2}\right)^3\right\}=$

$\frac{1}{2}\pi\times25\times(125-27)=49\times25\times3.14159265=$ 3848.45118.

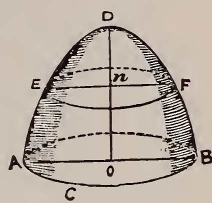


FIG. 52.

II.
1. 30= the radius AO of the base of the conoid.
2. 20=the altitude OD. Then by a property of the parabola,
3. $30^2=2p\times20$, whence
4. $p=22\frac{1}{2}$, the principal parameter of the parabola.
5. $\left(\frac{15}{2}\right)^3\times125=\sqrt{\left[(22\frac{1}{2})^2+30^2\right]^3}$=the square root of the cube of the sum of the squares of half the latus rectum and the radius of the base.
6. $\left(\frac{45}{2}\right)^3$=the cube of half the latus rectum.
7. $\left(\frac{15}{2}\right)^3\times125-\left(\frac{45}{2}\right)^3=\left(\frac{15}{2}\right)^3(125-27)$=the difference between said square root and the cube of half the latus rectum.
8. $2\pi\times\left(\frac{15}{2}\right)^3(125-27)=\pi\times98\times\left(\frac{15}{2}\right)^3=2\pi$ times said difference.
9. $\therefore 2\pi98\times\left(\frac{15}{2}\right)^3\div(\frac{3}{2}\times45)=3848.45118$=the surface of the conoid.

III. $\therefore$ The surface of the conoid is 3848.45118.

Prob. CXXI. To find the volume of a parabolic conoid.

Formula.— $V=\int\pi y^2dx=\int\pi2p\times dx=\pi px^2=\frac{1}{2}\pi(2px)x=\frac{1}{2}\pi y^2x$; where y is the radius of the base and x the altitude.

Rule.—*Multiply the area of the base by the altitude and take half the product.*

I. What is the volume of parabolic conoid, the radius of whose base is 10 feet and the altitude 14 feet?

By formula, $V=\frac{1}{2}\pi y^2x=\frac{1}{2}\pi 10^2\times 14=700\times\pi=2202.114855$ cu. ft.

II.
1. 10 ft.=the radius of the base.
2. 14 ft.=the altitude.
3. $\pi 10^2=314.159265$ sq. ft. the area of the base.
4. $\therefore\ \frac{1}{2}\times 14\times 314.159265=2202.114855$ cu. ft.=the volume of the conoid.

III. $\therefore$ The volume of the conoid=2202.114855 cu. ft.

NOTE.—Since the volume of the conoid is $\frac{1}{2}\pi y^2x$, it is half of its circumscribed cylinder.

Prob. CXXII. To find the convex surface of a frustum of a parabolic conoid of which the radius of the lower base is R and the upper base r.

$$\textit{Formula.}—S=\int_r^R 2\pi y ds=\frac{2\pi}{3p}\left\{(p^2+R^2)^{\frac{3}{2}}-(p^2+r^2)^{\frac{3}{2}}\right\}.$$

I. What is the volume of the frustum of a parabolic conoid of which the radius of the lower base is 12 feet, the radius of the upper base 8 feet, and the altitude of the frustum 5 feet?

Since $12^2=2px'$ and $8^2=2px$, $12^2-8^2=2p(x'-x)$. Bnt $x'-x=5$ feet. $\therefore\ 12^2-8^2=2p\times 5$, whence $2p=16$, the latus rectum.

$\therefore$ By formula, $S=\frac{2\pi}{3p}\left[(p^2+R^2)^{\frac{3}{2}}-(p^2+r^2)^{\frac{3}{2}}\right]=$
$\frac{2\pi}{3\times 8}\left[(8^2+12^2)^{\frac{3}{2}}-(8^2+8^2)^{\frac{3}{2}}\right]=\frac{\pi}{12}(832\sqrt{13}-1024\sqrt{2})=$
$\frac{16}{3}\pi[13\sqrt{13}-16\sqrt{2}]$.

Prob. CXXIII. To find the volume of the frustum of a parabolic conoid, when the bases are perpendicular to the axis of abscissa.

$$\textit{Formula.}— V=\frac{1}{2}\pi R^2x'-\frac{1}{2}\pi r^2x=\frac{1}{2}\pi(x'-x)(R^2+r^2)$$
$$\frac{1}{2}\pi a(R^2+r^2).$$

Rule.—*Multiply the sum of the squares of the radii of the two bases by* π *and this product by half the altitude.*

I. What is the volume of the frustum of a parabolic conoid, the diameter of the greater end being 60 feet, and that of the lesser end 48 feet, and the distance of the ends 18 feet?

By formula, $V=\frac{1}{2}\pi a(R^2+r^2)=\frac{1}{2}\times 18\times\pi(30^2+24^2)=9\pi(900+576)=9\times 1476\times\pi=13284\pi=41732.9177626$ cu. ft,

II.
1. 30 ft.=the radius of the larger base.
2. 24 ft.=the radius of the lesser base.
3. 18 ft.=the altitude of the frustum.
4. 900 sq. ft.=the square of the radius of the lower base.
5. 576 sq. ft.=the square of the radius of the upper base.
6. 1476 sq. ft =900 sq. ft.+576 sq. ft.=their sum.
7. $\therefore \frac{1}{2}\times18\times\pi\times1476=13284\times\pi=41732.9177626$ cu. ft.= the volume of the frustum of the conoid.

III. $\therefore$ The volume of the frustum is 41732.9177626 cu. ft.

II. THE HYPERBOLIC CONOID.

1. ***An Hyperbolic Conoid*** is the solid formed by revolving an hyperbola about its axis of abscissa.

Prob. CXXIV. To find the surface of an hyperbolic conoid, or hyperboloid.

Formula.—$S=\int 2\pi y ds=2\pi\int y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=$

$$2\pi\int y\sqrt{\frac{e^2x^2-a^2}{x^2-a^2}}dx=2\pi\int_a^b\sqrt{e^2x^2-a^2}\,dx=\pi\frac{b}{a}\left\{x\sqrt{e^2x^2-a^2}-\frac{a^2}{e}\log\left[x+\frac{1}{e}\sqrt{e^2x^2-a^2}\right]\right\}+C=\pi\frac{b}{a}\left\{x\sqrt{e^2x^2-a^2}-ab-\frac{a^3}{\sqrt{a^2+b^2}}\log\left[\frac{a+\frac{ab}{\sqrt{a^2+b^2}}}{x+\frac{1}{e}\sqrt{e^2x^2-a^2}}\right]\right\}.$$

Prob. CXXV. To find the volume of an hyperbolic conoid.

Formula.— $V=\frac{1}{6}\pi(R^2+d^2)h$, where R is the radius of the base, d the middle diameter, and h the altitude.

Rule.—*To the square of the radius of the base add the square of the middle diameter between the base and the vertex; and this sum multiplied by the altitude, and the product again by* $\frac{1}{6}\pi$, *will give the solidity.*

I. In the hyperboloid ACB, the altitude CO is 10, the radius AO of the base 12, and the middle diameter DE 15.8745; what is the volume?

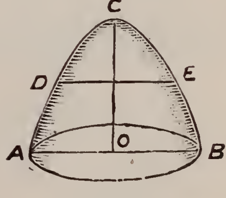


FIG. 53.

II.
1. 10=the altitude CO.
2. 12=the radius AO of the base.
3. 15.8745=the middle diameter DE.
4. $144=12^2$=the square of the radius of the base.
5. $251.99975=15.8745^2$=the square of the middle diameter.

6. 395.99975=251.99975+144=the sum of the squares of the radius of the base and the middle diameter.
7. $\therefore$ $\frac{1}{6}\pi\times10\times395.99975=2073.454691$=the volume.

III. $\therefore$ The volume of the conoid is 2073.454691.

Prob. CXXVI To find the volume of the frustum of an hyperbolic conoid.

Formula.— $V=\frac{1}{6}\pi a(R^2+d^2+r^2)$, where R is the radius of the larger base, and r the radius of the lesser base, and d the middle diameter of the frustum.

Rule.—*Add together the squares of the greater and lesser semi-diameters, and the square of the whole diameter in the middle; then this sum being multiplied by the altitude, and the product again by* $\frac{1}{6}\pi$, *will give the solidity.*

XV. QUADRATURE AND CUBATURE OF SURFACES AND SOLIDS OF REVOLUTION.

1. CYCLOID.

Prob. CXXVII. To find the surface generated by the revolution of a cycloid about its base.

Formula.—$S=2\int2\pi yds=4\pi\int y\sqrt{dx^2+dy^2}=$ $4\pi\sqrt{2r}\int_0^{2r}\frac{ydy}{\sqrt{2r-y}}=\frac{64}{3}\pi r^2$.

Rule.—*Multiply the area of the generating circle by* $\frac{64}{3}$.

Prob. CXXVIII. To find the volume of the solid formed by revolving the cycloid about its base.

Formula.— $V=2\int\pi y^2dx=2\pi\int_0^{2r}\frac{y^3dy}{\sqrt{2ry-y^2}}=5\pi^2r^3=$ $\frac{5}{8}\times\pi(2r)^2\times2\pi r$.

Rule.—*Multiply the cube of the radius of the generating circle by* $5\pi^2$.

Prob. CXXIX. To find the surface generated by revolving the cycloid about its axis.

Formula.—$S=\int2\pi yds=4\pi\sqrt{2r}\int y\frac{dx}{x}=8\pi r^2(\pi-\frac{4}{3})$.

Rule.—*Multiply eight times the area of the generating circle by* π *minus* $\frac{4}{3}$.

Prob. CXXX. To find the volume of the solid formed by revolving the cycloid about its axis.

Formulu.— $V=\int \pi y^2 dx = 2\pi \int y^{\frac{5}{2}}(2r-y)^{-\frac{1}{2}}dy=$ $\pi r^3(\frac{3}{2}\pi^2-\frac{8}{3})$.

Rule.—*Multiply* $\frac{3}{4}$ *of the volume of a sphere whose radius is that of the generating circle by* $\frac{3}{2}\pi^2-\frac{8}{3}$.

Prob. CXXXI. To find the surface formed by revolving the cycloid about a tangent at the vertex.

Let P be a point on the curve, $AE=PB=y$, $EP=AB=x$, $AC=CF=r$, and the angle $ACF=\theta$. Then we shall have $AE=y=AC-CE=r-r\cos\theta$; and $AB=x=FP+EF=AF+EF=r\theta+r\sin\theta$.

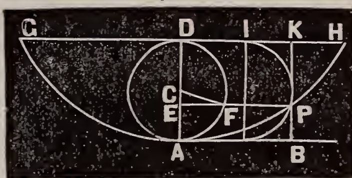


FIG. 54.

$\therefore$ ***Formula.***—$S=$

$$4\pi\int y\sqrt{dx^2+dy^2}=4\pi r\int_0^\pi(r-r\cos\theta)\sqrt{r^2(1+\cos\theta)^2+r^2\sin^2\theta}\,d\theta=4\pi r^2\int_0^\pi(1-\cos\theta)\times$$

$$\sqrt{2+2\cos\theta}\,d\theta=8\pi r^2\int_0^\pi(1-\cos\theta)\cos\tfrac{1}{2}\theta\,d\theta=16\pi r^2\int_0^\pi(1-\cos^2\tfrac{1}{2}\theta)$$

$$\times\cos\tfrac{1}{2}\theta\,d\theta=16\pi r^2\int_0^\pi(\cos\tfrac{1}{2}\theta-\cos^3\tfrac{1}{2}\theta)d\theta=16\pi r^2\Big[2\sin\tfrac{1}{2}\theta-$$

$$\tfrac{2}{3}\sin\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta-\tfrac{4}{3}\sin\tfrac{1}{2}\theta\Big]_0^\pi=\tfrac{32}{3}\pi r^2.$$

Rule.—*Multiply the area of the generating circle by* $\frac{32}{3}$.

Prob. CXXXII To find the volume formed by revolving a cycloid about a tangent at the vertex.

Formula.— $V'=2\int\pi y^2\,dx=2\pi\int_0^\pi(r-r\cos\theta)^2 r(1+\cos\theta)d\theta=2\pi r^3\int_0^\pi(1-\cos\theta)^2(1+\cos\theta)d\theta=2\pi r^3\int_0^\pi(1-\cos\theta-\cos^2\theta+\cos^3\theta)d\theta=\pi^2r^3=$the volume generated between the curve and the tangent.

$\therefore\ V=\pi AD^2\times GH-V'=\pi(2r)^2\times 2\pi r-\pi^2r^3=7\pi^2r^3$.

Rule.—*Multiply the cube of the radius of the generating circle by* $7\pi^2$.

2. CISSOID.

Prob. CXXXIII. To find the volume generated by revolving the cissoid about the axis of abscissa.

Formula.— $V=\int\pi y^2dx=\int\pi\frac{x^3}{2a-x}dx=\pi\left(-\tfrac{1}{3}x^3-ax^2-4a^2x+8a^3\log\left(\frac{2a}{2a-x}\right)\right)$.

Prob. CXXXIV. To find the volume formed by revolving the cissoid about its asymptote.

Formula.— $V=2\int\pi(AR)^2dy(Fig.\ 22)=2\pi(2a-x)\times\frac{(3a-x)\sqrt{x}}{(2a-x)^{\frac{3}{2}}}dx=2\pi[\frac{1}{3}(2ax-x^2)^{\frac{3}{2}}+2a\int(2ax-x^2)dx]=2\pi^2a^3.$

Prob. CXXXV. To find the volume formed by revolving the Witch of Agnesi about its asymptote.

Formula.— $V=\int\pi y^2dx=\Big[\pi y^2x-4\pi a\int(2ay-y2)^{\frac{1}{2}}dy\Big]_0^{2a}=4\pi^2a^3$

Prob. CXXXVI. To find the volume formed by revolving the Conchoid of Nicomedes about its asymptote, or axis of abscissa.

Formula.— $V=\int\pi y^2dx=\pi\int\Big[-\frac{ab^2}{\sqrt{b^2-y^2}}+y(b^2-y^2)^{\frac{1}{2}}-b^2\frac{y}{\sqrt{b^2-y^2}}\Big]dy=\pi b^2\left(\pi a+\frac{4b}{3}\right).$

2. SPINDLES.

A Circular Spindle is the solid formed by revolving the segment of a circle about its chord.

Prob. CXXXVII. To find the volume of a circular spindle.

FIG. 55.

Let $AEBD$ be the circular spindle formed by revolving the segment $ACBE$ about the chord ACB. Let $AB=2a$, the length of the spindle, and $ED=2b$, the middle diameter of the spindle. Let $CI=KL=x$, the radius of any right section of the spindle, and $KI=CL=y$. Then the required volume of the spindle is $V=2\pi\int_0^a x^2dy$...(1). Let $R=(a^2+b^2)\div2b$..(2), be the radius of the circle and θ the angle AGE. Then by a property of the circle, $KI^2=(2R-EI)\times EI$, or $y^2=(2R-EI)\times EI$. But $EI=EG-IG=R-(IC+CG)=R-(x+R\cos\theta)$. $\therefore y^2=\{2R-[R-(x+R\cos\theta)\}\{R-(x+R\cos\theta)\}$ $=[R+(R\cos\theta+x)]\times[R-(R\cos\theta+x)]=R^2-(R\cos\theta+x)^2$; whence $x=\sqrt{R^2-y^2}-R\cos\theta$ ..(3). Substituting this value of x in (1), we have $V=2\pi\int_0^a(\sqrt{R^2-y^2}-R\cos\theta)^2dy=2\pi[R^2(1+$

$$\cos^2\theta)y-\tfrac{1}{3}y^3-2R\cos\theta\left(\tfrac{1}{2}R^2\sin^{-1}\frac{y}{R}-\tfrac{1}{2}y\sqrt{R^2-y^2}\right)\Big]_0^a=2\pi\Big\{\tfrac{2}{3}a^3$$
$$-(R-b)\left[R^2\sin^{-1}\frac{a}{R}-a\sqrt{R^2-a^2}\right]\Big\}.$$

Rule.—*Multiply the area of the generating segment by the path of its center of gravity.—Guildin's Rule.*

3. THE PARABOLIC SPINDLE.

A Parabolic Spindle is a solid formed by revolving a parabola about a double ordinate perpendicular to the axis.

Prob. CXXXVIII. To find the volume of a parabolic spindle.

Formula.— $V=2\int_0^b \pi(h-x)^2dy=2\pi\int_0^b(h^2-2hx+x^2)dy=2\pi\int_0^b\left(h^2\,dy-2h\frac{y^2}{2p}dy+\frac{y^4}{4p^2}dy\right)=2\pi\left[h^2y-\tfrac{1}{3}\frac{h}{p}y^3+\tfrac{1}{20}\frac{1}{p^2}y^5\right]_0^b=2\pi\left[h^2y-\tfrac{2}{3}hxy+\tfrac{1}{5}x^2y\right]_0^b=2\pi]h^2b-\tfrac{2}{3}hbx+\tfrac{1}{5}bx^2]$. But $x=h$, when $y=b$.

$\therefore\ V=2\pi[h^2b-\tfrac{2}{3}h^2b+\tfrac{1}{5}h^2b]=\tfrac{16}{15}\pi h^2b=\tfrac{8}{15}\times 2b\times\pi h^2.$

Rule.—*Multply the volume of its circumscribed cylinder by $\tfrac{8}{15}$.*

I. What is the volume of a parabolic spindle whose length AC is 3 feet and height BD 1 foot?

By formula, $V=\tfrac{16}{15}\pi h^2b=\tfrac{8}{15}\pi\times1^2\times3=4.9945484$ cu. ft.

II.
1. 1 ft.=height BD of the spindle.
2. 3 ft.=length AC.
3. $\pi\times1^2\times3=9.42477795$ cu. ft. the volume of its circumscribed cylinder.
4. $\therefore$ $\tfrac{8}{15}\times9.42477795$ cu. ft.= 4.9945484 cu. ft., the volume of the parabolic spindle.

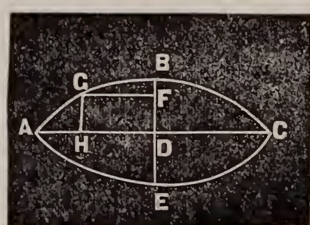


FIG. 56.

III. $\therefore$ The volume of the spindle is 4.9945484 cu. ft.

Prob. CXXXIX. To find the volume generated by revolving the arc of a parabola about the tangent at its vertex.

Let APC be an arc of a parabola revolved about AB, and let P be any point of the curve. Let $AE=PF=x$, and $AF=PF=y$. Then the area of the circle described by the line PF is πx^2.

$\therefore$ ***Formula.***— $V=2\pi\int x^2dy=2\pi\int\frac{y^4}{4p^2}dy=2\pi\times\frac{1}{3p^2}\times$

$\frac{1}{5}y^5=\frac{1}{5}\pi x^2 y=\frac{1}{5}\pi h^2 b$, where h=the height, and $b=CD$, the ordinate of the curve.

Rule.—*Multiply the volume of its circumscribed cylinder by $\frac{1}{5}$.*

Prob. CXL. To find the volume generated by revolving the arc APC of the parabola about BC parallel to the axis AD.

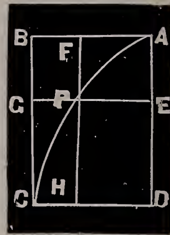


FIG. 57.

The area of the circle generated by the line GP is $\pi(b-y)^2$.

$\therefore$ ***Formula.***— $V=\pi\int(b-y)^2dx=\frac{1}{6}\pi b^2 h$.

Rule.—*Multiply the volume of its circumscribed cylinder by $\frac{1}{6}$.*

NOTE.—In the last two problems, the volume considered, lies between the curve and the lines AB and BC respectively. The volume generated by the segment ACD is found by subtracting the volume found in the two problems from the volume of the circumscribed cylinders.

Prob. CXLI. To find the volume formed by revolving a semi-circle about a tangent parallel to its diameter.

FIG. 59.

Let the semi-circle be revolved about the tangent AC. Let $AC=R, PF=AG=EC=y, AF=GP=x$. Then the area of the circle generated by the line GP is πx^2. But $x^2=2R^2-2R(R^2-y^2)^{\frac{1}{2}}-y^2$; for, $FC^2=PC^2-PF^2$, or $(R-x)^2=R^2-y^2$; whence $x=R-\sqrt{R^2-y^2}$, and $x^2=2R^2-2R\sqrt{R^2-y^2}-y^2$

$\therefore$ ***Formula.***— $V=2\int\pi x^2dy=2\pi\int(2R^2-2R\sqrt{R^2-y^2}-y^2)dy=\frac{1}{3}\pi R^3(10-3\pi)$, which is the entire volume external to the semi-circle.

Rule.—*Multiply one-fourth of the volume of a sphere whose radius is that of the generating semi-circle by $(10-3\pi)$.*

XVI. REGULAR SOLIDS.

1. ***A Regular Solid*** is a solid contained under a certain number of similar and equal plane figures.

2. ***The Tetrahedron,*** or ***Regular Pyramid,*** is a regular solid bounded by four triangular faces.

3. ***The Hexahedron,*** or ***Cube,*** is a regular solid bounded by six square faces.

4. ***The Octahedron*** is a regular solid bounded by eight triangular faces.

5. ***The Dodecahedron*** is a regular solid bounded by twelve pentagonal faces.

6. ***The Icosahedron*** is a regular solid bounded by twenty equilateral triangular faces.

These are the only regular solids that can possibly be formed.

If the following figures are made of pasteboard, and the dotted lines cut half through, so that the parts may be turned up and glued together, they will represent the five regular solids.

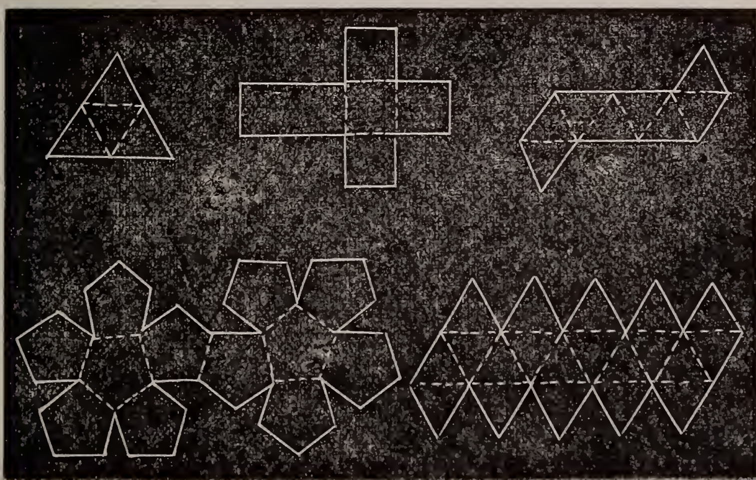


FIG. 59.

1. TETRAHEDRON.

Prob. CXLII. To find the surface of a tetrahedron.

Formula.—$S=l^2\sqrt{3}$, where l is the length of a linear side.

Rule.—*Multiply the square of a linear side by $\sqrt{3}=1.7320508$.*

I. What is the surface of a tetrahedron whose linear edge is 2 inches.

By formula, $S=l^2\sqrt{3}=2^2\sqrt{3}=4\sqrt{3}=6.9282$ sq. in.

II. 1. 2 in.=the length of a linear side.
2. 4 sq. in.$=2^2$=the square of a linear side.
3. ∴ $\sqrt{3}\times 4$ sq. in.$=1.73205\times 4$ sq. in.=6.9282 sq. in., the surface.

III. ∴ The surface of the tetrahedron is 6.9282 sq. in.

Prob. CXLIII. To find the volume of a tetrahedron.

Formula.—$V=\frac{1}{12}\sqrt{2}\, l^3$, where l is the length of a linear side.

Rule.—*Multiply the cube of a linear side by* $\frac{1}{12}\sqrt{2}$, *or* .11785.

I. Required the solidity of a tetrahedron whose linear side is 6 feet?

By formula, $V=\frac{1}{12}\sqrt{2}\,l^3=\frac{1}{12}\sqrt{2}\times6^3=18\sqrt{2}=25.455843$ cu. ft.

II. 1. 6 ft.=the length of a linear side.
2. 216 cu. ft.=the cube of the linear side.
3. $\therefore$ $\frac{1}{12}\sqrt{2}\times216$ cu. ft.$=\sqrt{2}\times18$ cu. ft.=25.45843 cu. ft.

III. $\therefore$ The volume of the tetrahedron is 25.45843 cu. ft.

2. OCTAHEDRON.

Prob. CXLIV. To find the surface of an octahedron.

Formula.—$S=2\sqrt{3}\,l^2$.

Rule.—*Multiply the square of a linear side by* $2\sqrt{3}$, *i. e., by two times the square root of three.*

I. What is the surface of an octahedron whose linear side is 4 feet?

By forumla, $S=2\sqrt{3}\,l^2=2\sqrt{3}\times l^2=32\sqrt{3}=55.4256$ cu. ft.

II. 1. 4 ft.=the length of a linear side.
2. 16 sq. ft.$=4^2$=the square of the linear side.
3. $\therefore$ $2\sqrt{3}\times16$ sq. ft.$=\sqrt{3}\times32$ sq. ft.$=1.73205\times32$ sq. ft.= 55.4256 sq. ft.

III. $\therefore$ The surface of the octahedron is 55.4256 sq. ft.

Prob. CLXV. To find the volume of an octahedron.

Formula.—$V=\frac{1}{3}\sqrt{2}\,l^3$

Rule.—*Multiply the cube of a linear side by* $\frac{1}{3}\sqrt{2}$, *i. e., by one-third of the square root of two.*

I. What is the volume of an octahedron whose linear side is 8 inches?

By formula, $V=\frac{1}{3}\sqrt{2}\,l^3=\frac{1}{3}\sqrt{2}\times8^3=.4714045\times512=241.359104$ cu. in.

II. 1. 8 in.=the length of a linear side.
2. 512 cu. in.$=8^3$=the cube of a linear side.
3. $\therefore$ $\frac{1}{3}\sqrt{2}\times512$ cu. in.$=\frac{1}{3}\times1.4142135\times512$ cu. in.= 241.359104 cu. in.

III. $\therefore$ The volume of the octahedron is 241.359104 cu. iu.

3. DODECAHEDRON.

Prob. CXLVI. To find the surface of a dodecahedron.

Formula.—$S=15\sqrt{\left(\frac{5+2\sqrt{5}}{5}\right)}l^2=20.6457285\times l^2$.

Rule.—*Multiply the square of a linear side by* $15\sqrt{[\frac{1}{5}(5+2\sqrt{5})]}$, *or 20.6457285.*

I. What is the surface of a dodecahedron whose linear side is 3 feet?

By formula, $S=15\sqrt{\left(\frac{5+2\sqrt{5}}{5}\right)}l^2=20.6457285\times 9$
$=185.8115565$ sq. ft.

II.
1. 3 ft.=the length of a linear side.
2. 9 sq. ft.$=3^2$=square of a linear side.
3. $\therefore 15\sqrt{\left(\frac{5+2\sqrt{5}}{5}\right)}\times 9$ sq. ft$=20.6457285\times 9$ sq. ft.
$=185.8115565$ sq. ft.

III. The surface of the dodecahedron is 185.8115565 sq. ft.

Prob. CXLVII. To find the volume of a dodecahedron.

Formula.—$V=5\sqrt{\left(\frac{47+21\sqrt{5}}{40}\right)}l^3=7.663115\times l^3$.

Rule.—*Multiply the cube of a linear side by* $5\sqrt{\left(\frac{47+21\sqrt{5}}{40}\right)}$, *or 7.663115.*

I. The linear side of a dodecahedron is 2 feet; what is its volume?

By formula, $V=5\sqrt{\left(\frac{47+21\sqrt{5}}{40}\right)}l^3=7.663115\times 8$
$=61.20492$ cu. ft.

II.
1. 2 ft.=the length of a linear side.
2. 8 cu. ft.$=2^2$=cube of a linear side.
3. $\therefore 5\sqrt{[\frac{1}{40}(47+21\sqrt{5})]}\times 8$ cu. ft.$=7.663115\times 8$ cu. ft.
$=61.20492$ cu. ft., the volume.

III. $\therefore$ The volume of the dodecahedron is 61.20492 cu. ft.

4. ICOSAHEDRON.

Prob. CXLVIII. To find the surface of an icosahedron.

Formula.—$S=5\sqrt{3}l^2=8.66025\times l^2$.

Rule.—*Multiply the square of a linear side by* $5\sqrt{3}$, *or 8.66025.*

I. What is the surface of an icosahedron whose linear side is 5 feet.

By formula, $S=5\sqrt{3}l^2=5\sqrt{3}\times5^2=125\sqrt{3}=216.50625$ sq. ft.

II. $\begin{cases} 1.\ 5\text{ ft.}=\text{length of a linear side.} \\ 2.\ 25\text{ sq. ft.}=5^2=\text{the square of a linear side.} \\ 3.\ \therefore 5\sqrt{3}\times25\text{ sq. ft.}=8.66025\times25\text{ sq. ft.}=216.50625\text{ sq. ft.} \\ \quad =\text{the surface.} \end{cases}$

III. $\therefore$ The surface of the icosahedron is 216.50625 sq. ft.

Prob. CXLIX. To find the solidity of an icosahedron.

Formula.— $V=\frac{5}{6}\sqrt{[\frac{1}{2}(7+3\sqrt{5})]}l^3=2.18169\times l^3$.

Rule.—*Multiply the cube of a linear side by $\frac{5}{6}\sqrt{[\frac{1}{2}(7+3\sqrt{5})]}$, or 2.18169*

I. What is the volume of an icosahedron whose linear side is 3 feet?

By formula, $V=\frac{5}{6}\sqrt{[\frac{1}{2}(7+3\sqrt{5})]}l^3=2.18169\times3^3=58.90563$ cu.ft.

II. $\begin{cases} 1.\ 3\text{ ft.}=\text{the length of a linear side.} \\ 2.\ 27\text{ cu. ft.}=3^3=\text{the cube of a linear side.} \\ 3.\ \therefore \frac{5}{6}\sqrt{[\frac{1}{2}(7+3\sqrt{5})]}\times27\text{ cu. ft.}=21.8169\times27\text{ cu. ft.} \\ \quad =58.90563\text{ cu. ft.}=\text{the volume.} \end{cases}$

III. $\therefore$ The volume of the icosahedron is 58.90563 cu. ft.

Note.—The surface and volume of any of the five regular solids may be found as follows:

Rule (1).—*Multiply the tabular area by the square of a linear side, and the product will be the surface*

Rule (2).—*Multiply the tabular volume by the cube of a linear side, and the product will be the volume.*

Surfaces and volumes of the regular solids, the edge being 1.

NO. OF SIDES.	NAMES.	SURFACES.	VOLUMES.
4	Tetrahedron	1.73205	0.11785
6	Hexahedron	6.00000	1.00000
8	Octahedron	3.46410	0.47140
12	Dodecahedron	20.64578	7.66312
20	Icosahedron	8.66025	2.18169

XVII. PRISMATOID.

1. ***A Prismatoid*** is a polyhedron whose bases are any two polygons in parallel planes, and whose lateral faces are triangles determined by so joining the vertices of these bases, that each lateral edge, with the preceding, forms a triangle with one side of either base.

2. ***A Prismoid*** is a prismatoid whose bases have the same number of sides, and every corresponding pair parallel.

Prob. CL. To find the volume of any prismatoid.

Formula (*a*).— $V=\frac{1}{4}a(B_1+3A_{\frac{2}{3}a})=\frac{1}{4}a(B_2+3A'_{\frac{2}{3}a})$, where a is the altitude, B_1 the area of the lower base, $A_{\frac{2}{3}a}$ the area of a section distant from the lower base two-thirds the altitude, B_2 the area of the upper base, and $A'_{\frac{2}{3}a}$ the area of a section distant two-thirds the altitude from the upper base.

Remark.—This simplest Prismoidal Formula is due to Prof. George B. Halstead, A. M., Ph. D., Professor of Mathematics in the University of Texas, Austin, Texas, who was the first to demonstrate this important truth. The formula universally applies to all prisms and cylinders; also to all solids uniformly twisted, e. g. the square screw; also to the paraboloid, the right circular cone, the frustum of a paraboloid, the hyperboloid of one nappe, the sphere, prolate spheroid, oblate spheroid, frustum of a right cone, or of a sphere, spheroid, or the elliptic paraboloid, the groin, hyperboloid, or their frustums. For a complete demonstration of the Prismoidal Formula, see *Halstead's Elements of Geometry or Halstead's Mensuration.*

Rule.—(*a*) *Multiply one-fourth its altitude by the sum of one base and three times a section distant from that base two-thirds the altitude.*

Formula.— $V=\frac{1}{6}a(B_1+4M+B_2)$, where a is the altitude, B_1 and B_2 the areas of the lower and upper bases respectively, and M the area of a section midway between the two bases.

Rule.—(*b*) *Add the area of the two bases and four times the mid cross-section; multiply this sum by one-sixth the altitude.*

XVIII. CYLINDRIC RINGS.

1. ***A Cylindric Ring*** is a solid generated by a circle lying wholly on the same side of a line in its own plane and revolving abont that line. Thus, if a circle whose center is O be revolved about DC as an axis, it will generate a cylindric ring whose diameter is AB and inner diameter $2\ BC$. OC will be the radius of the path of the center O.

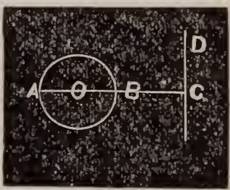


FIG. 60.

Prob. CLI. To find the area of the surface of a solid ring.

Formula.—$S=2\pi r\times 2\pi R=4\pi^2 rR$, whare r is the radius of the ring, and R is the distance from the center of the ring to the center of the inclosed space.

Rule.—*Multiply the generating circumference by the path of its center. Or, to the thickness of the ring add the inner diameter and this sum being multiplied by the thickness, and the product again by 9.8697044 will give the area of the surface.*

I. What is the area of the surface of a ring whose diameter is 3 inches and the inner diameter 12 inches.

By formula, $S=4\pi^2rR=4\pi^2\times1\frac{1}{2}\times(1\frac{1}{2}+6)=\pi^2\times45=9.8696044\times45=444.132198$ sq. in.

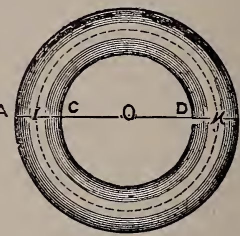


FIG. 61.

II.
1. $1\frac{1}{2}$ in. $=\frac{1}{2}$ of 3 in. = the radius r of the ring.
2. 6 in. $=\frac{1}{2}$ of 12 in. = the radius of the inclosed space.
3. 6 in. $+1\frac{1}{2}$ in. $=7\frac{1}{2}$ in. = the radius R of the center of the ring.
4. $\pi AC=\pi3$ = the circumference of a section.
5. $\pi IK=2\pi IO=2\pi7\frac{1}{2}=\pi15$ = the path of the center.
6. $\therefore$ $\pi3\times\pi15=\pi^245=444.132198$ sq. in. = the area of the surface of the ring.

III. $\therefore$ The area of the surface of the ring is 444.132198 sq. in.

Prob. CLII. To find the volume of a cylindric ring.

Formula.— $V=\pi^2r^2R=\pi r^2\times\pi R$, where r is the radius AI of the ring, and R the distance from the center of the ring to the center of the inclosed space.

Rule.—*Multiply the area of the generating circle by the path of its center. Or, to the thickness of the ring add the inner diameter, and this sum being multiplied by the square of half the thickness, and the product again by 9.8696044, will give the volume.*

I. What is the volume of an anchorring whose inner diameter is 8 inches, and thickness in metal 3 inches?

By formula, $V=\pi^2r^2R=\pi^2\times(1\frac{1}{2})^2\times(3+8)=24.75\times9.8696044=244.2727089$ cu. in.

II.
1. $1\frac{1}{2}$ in. $=\frac{1}{2}$ of 3 in. = the radius of the ring.
2. 8 in. = the inner diameter.
3. 4 in. $+1\frac{1}{2}$ in. $=5\frac{1}{2}$ in. = the radius R of the path of its center.
4. $\pi(1\frac{1}{2})^2$ = the area of the generating circle.
5. $2\pi(5\frac{1}{2})=\pi\times11$ = the path of its center.
6. $\therefore$ $\pi11\times\pi(1\frac{1}{2})^2=\pi^2\times24.75=9.86044\times24.75=244.2727089$ cu. in., the volume of the ring.

III. $\therefore$ The volume of the ring is 244.2727089 cu. in.

THEOREM OF PAPPUS.

If a plane curve lies wholly on one side of a line in its own plane, and revolving about that line as an axis, it generates thereby a surface of revolution, the area of which is equal to the product of the length of the revolving line into the path of its center of mass; and a solid the volume of which is equal to the revolving area into the length of the path described by its center of mass.

XIX. MISCELLANEOUS MEASUREMENTS.

1. MASONS' AND BRICKLAYERS' WORK.

Masons' work is sometimes measured by the cubic foot, and sometimes by the *perch*. A perch is $16\frac{1}{2}$ ft. long, $1\frac{1}{2}$ ft. wide, 1 ft. deep, and contains $16\frac{1}{2}\times1\frac{1}{2}\times1=24\frac{3}{4}$ cu. ft.

Prob. CLIII. To find the number of perch in a piece of masonry.

Rule.—*Find the solidity of the wall in cubic feet by the rules given for the mensuration of solids, and divide the product by $24\frac{3}{4}$.*

I. What is the cost of laying a wall 20 feet long, 7 ft. 9 in. high, and 2 feet thick, at 75 cts. a perch.

II.
1. 20 ft.=the length of the wall,
2. 7 ft. 9 in.$=7\frac{3}{4}$ ft.=the height of the wall, and
3. 2 ft.=the thickness.
4. $\therefore 20\times7\frac{3}{4}\times2=310$ cu. ft.=the solidity of the wall.
5. $24\frac{3}{4}$ cu. ft.=1 perch.
6. 310 cu. ft.$=310\div24\frac{3}{4}=12\frac{52}{99}$ perches.
7. 75 cts.=the cost of laying 1 perch.
8. $\therefore 12\frac{52}{99}\times75$cts.$=\$9.39\frac{13}{33}$=the cost of laying $12\frac{52}{99}$ perches.

III. $\therefore$ It will cost $\$9.39\frac{13}{33}$ to lay $12\frac{52}{99}$ perches at 75 cts. a perch.

2. GUAGING.

Gauging is finding the contents of a vessel, in bushels, gallons, or barrels.

Prob. CLIV. To gauge any vessel.

Rule.—*Find its solidity in cubic feet by rules already given; this multiplied by $1728\div2150.42$ or .83, will give the contents in bushels; by $1728\div231$. will give it in wine gallons, which divided by $31\frac{1}{2}$ will give the contents in barrels.*

Prob. CLV. To find the contents in gallons of a cask or barrel.

Rule.—(*1*) *When the staves are straight from the bung to each end; consider the cask two equal frustums of equal cones, and find its contents by the rule of Prob. XCIII.*

(2). *When the staves are curved; Add to the head diameter (inside) two-tenths of the difference between the head and bung diameter; but if the staves are only slightly curved, add six-tenths of this difference; this gives the mean diameter; express it in inches, square it, multiply it by the length in inches, and this product by .0034; the product will be the contents in wine gallons.*

3. LUMBER MEASURE.

Prob. CLVI. To find the amount of square-edged inch boards that can be sawed from a round log.

Doyle's Rule.—*From the diameter in inches subtract four; the square of the remainder will be the number of square feet of inch boards yielded by a log 16 feet long.*

I. How much square-edged inch lumber can be cut from a log 32 in. in diameter, and 12 feet long?

II.
1. 32 in.=the diameter of the log.
2. 12 ft.=the length.
3. 32 in.—4 in.=28 in.=the diameter less 4.
4. 844 ft.=28^2=the square of the diameter less 4, which by the rule, is the number of feet in a log 16 ft. long.
5. 12 ft.=$\frac{3}{4}$ of 16 ft.
6. ∴ $\frac{3}{4}$ of 844 ft.=633 ft.=the number of feet of square-edged inch lumber that can be cut from the log.

III. ∴ The number of square-edged inch lumber that can be cut from a round log 32 inches in diameter and 12 ft. long is 633 ft.

4. GRAIN AND HAY.

Prob. CLVII. To find the quantity of grain in a wagon bed or in a bin.

Rule.—*Multiply the contents in cubic feet by 1728÷2150.42, or .83.*

I. How many bushels of shelled corn in a bin 40 feet long, 16 feet wide and 10 feet high?

II.
1. 40 ft.=the length of the bin.
2. 16 ft.=the width of the bin, and
3. 10 ft.=the height of the bin.
4. ∴ $40\times16\times10$=6400 cu. ft.=the contents of the bin in cu. ft..
5. ∴ $6400\times.83$ bu.=5312 bu.=the contents of the bin in bu.

III. ∴ The bin will hold 5312 bu. of shelled corn.

Rule.—(*1*) *For corn on the cob, deduct one-half for cob.*

(*2*) *For corn not "shucked" deduct two-thirds for cob and shuck.*

I. How many bushels of corn on the cob will a wagon bed hold that is $10\frac{1}{2}$ feet long, $3\frac{1}{2}$ feet wide, and 2 feet deep?

II.
1. $10\frac{1}{2}$ ft.=the length of the wagon bed,
2. $3\frac{1}{2}$ ft.=its width, and
3. 2 ft.=its depth.
4. ∴ $10\frac{1}{2}\times3\frac{1}{2}\times2=73\frac{1}{2}$ cu. ft.=contents of the wagon bed [in cu. ft
5. ∴ $73\frac{1}{2}\times.8$ bu=58.8 bu.=number of bushels of shelled corn the bed will hold.
6. ∴ $\frac{1}{2}$ of 58.8 bu=29.4 bu.=the number of bushels of corn on the cob that it will hold.

III. ∴ The wagon bed will hold 29.4 bu. of corn on the cob.

Prob. CLVIII. To find the quantity of hay in a stack, rick, or mow.

Rule.—*Divide the cubical contents in feet by 550 for clover or by 450 for timothy; the quotient will be the number of tons.*

Prob. CLXIX. To find the volume of any irregular solid.

Rule.—*Immerse the solid in a vessel of water and determine the quantity of water displaced.*

I A being curious to know the solid contents of a brush pile, put the brush into a vat 16 feet long, 10 feet wide, and 8 feet deep and containing 5 feet of water. He found, after putting in the brush, that the water rose $1\frac{1}{2}$ feet; what was the contents of the brush pile?

II.
1. 16 ft.=the length of the vat,
2. 10 ft.=the width, and
3. $1\frac{1}{2}$ ft.=the depth to which the water rose.
4. ∴ $16\times10\times1\frac{1}{2}=240$ cu. ft.=the volume of the brush pile.

III. ∴ 240 cu. ft=the volume of the brush pile.

XX. SOLUTIONS TO MISCELLANEOUS PROBLEMS.

Prob. CLX. To find at what distance from either end, a trapezoid must be cut in two to have equal areas, the dividing line being parallel to the parallel sides.

Formula.—$d=A\div[\sqrt{\frac{1}{2}(b^2+b_1^2)}+b]=\frac{1}{2}(b+b_1)a\div[\sqrt{\frac{1}{2}(b^2+b_1^2)}+b]$, where A is the area of the trapezoid, b the lower base, and b_1, the upper base. $\sqrt{\frac{1}{2}(b^2+b_1^2)}$ is the length of the dividing line.

Rule.—*1. Extract the square root of half the sum of the squares of the parallel sides and the result will be the length of the dividing line.*

2. *Divide half the area of the whole trapezoid by half the sum of the dividing line and either end, and the quotient will be the distance of the dividing line from that end.*

I. I have an inch board 5 feet long, 17 inches wide at one end and 7 inches at the other; how far from the large end must it be cut straight across so that the two parts shall be equal?

FIG. 62.

By formula, $d=\frac{1}{2}(b+b_1)a\div[\sqrt{\frac{1}{2}(b^2+b_1^2)}+b]$ $=\frac{1}{2}(17+7)60\div[\sqrt{\frac{1}{2}(17^2+7^2)}+17]=720\div30$ $=24$ in.$=2$ ft.

II.
1. Let $ABCD$ be the board,
2. $AB=17$ in.$=b$, the width of the large end,
3. $DC=7$ in.$=b'$, the width of the small end, and
4. $HK=5$ ft.$=60$ in.$=a$, the length of the board.
5. Produce HK, AD, and BC till they meet in E. Then by similar triangles,
6. $ABE:EGL:EDC::AB^2:LG^2:DC^2$. But
7. $EGL=EDC+\frac{1}{2}(ABCD)$, or
8. $2EGL=2EDC+ABCD=EDC+EDC+ABCD$ $=EDC+EAB$.
9. $\therefore$ $EGL=\frac{1}{2}(EDC+EAB)$, *i. e.*, EGL is an arithmetic mean between EAB and EDC.
10. $\therefore$ $GL^2=\frac{1}{2}(AB^2+DC^2)=\frac{1}{2}(b^2+b'^2)=$an arithmetic mean between EAB and EDC,
11. $GL=\sqrt{\frac{1}{2}(b^2+b'^2)}=\frac{1}{2}\sqrt{2(b^2+b'^2)}$.
12. Draw CM perpendicular to AB.
13. $FL=\frac{1}{2}GL=\frac{1}{4}\sqrt{2(b^2+b'^2)}$.
14. $IL=FL-FI(=KC=\frac{1}{2}DC=\frac{1}{2}b)=\frac{1}{4}\sqrt{2(b^2+b'^2)}-\frac{1}{2}b$.
15. $CM=HK=a$.
16. $MB=\frac{1}{2}(b-b')$. Then in the similar triangles CMB and CIL,
17. $MB:IL::CM:CI$, or $\frac{1}{2}(b-b'):(\frac{1}{4}\sqrt{2(b^2+b'^2)}-\frac{1}{2}b)::a:CI$. Whence
18. $CI=a(\frac{1}{4}\sqrt{2(b^2+b'^2)}-\frac{1}{2}b)\div\frac{1}{2}(b-b')=$ $\dfrac{a(\frac{1}{2}\sqrt{2(b^2+b'^2)}-b)}{b-b'}=60\dfrac{(\frac{1}{2}\sqrt{2(17^2+7^2)}-7)}{17-7}=36$ in. $=3$ ft.
19. $\therefore$ $IM=CM-CI=5$ ft.-3 ft.$=2$ ft., the distance from the large end at which the board must be cut in two to have equal areas.

III. $\therefore$ The board must be cut in two, at a distance of 2 feet from the large end, to have equal areas in both parts.

(*R. H. A., p. 407, prob. 101.*)

Prob. CLXI. To divide a trapezoid into n equal parts and find the length of each part.

$$\textit{Formula.}—h_1=\frac{a}{b-b'}\left[\sqrt{\frac{(n-1)\,b'^2+b^2}{n}}-b\right],$$

$$h_2=\frac{a}{b-b'}\left[\sqrt{\left(\frac{(n-2)b'^2+2b^2}{n}\right)}-\sqrt{\left(\frac{(n-1)b'^2+b^2}{n}\right)}\right],$$

$$h_3=\frac{a}{b-b'}\left[\sqrt{\left(\frac{(n-3)b'^2+3b^2}{n}\right)}-\sqrt{\left(\frac{(n-2)b'^2+2b^2}{n}\right)}\right],\ \cdots$$

$$h_n=\frac{a}{b-b'}\left[b-\sqrt{\left(\frac{[n-(n-1)]b'^2+(n-1)b^2}{n}\right)}\right],$$ where b' is the width of the small end, b the width of the large end, and a the length of the trapezoid. h_1 is the length of the first part at the small end, h_2 the length of the second part, and so on.

I. A board $ABCD$ whose length BC is 36 inches, width AB 8 inches and DC 4 inches, is divided into three equal pieces. Find the length of each piece.

By formula, $h_1=\frac{a}{b-b_1}\left[\sqrt{\frac{(n-1)b_1^2+b^2}{n}}-b_1\right]=$

$\frac{36}{8-4}[\sqrt{\tfrac{1}{3}(3-1)4^2+8^2}-4]=9[\sqrt{32}-4]=36(\sqrt{2}-1)=14.911686$ in.

$h_2=\frac{a}{b-b_1}\left[\sqrt{\frac{(n-2)b_1^2+2b^2}{n}}-\sqrt{\frac{(n-1)b_1^2+2b^2}{n}}\right]=36(\sqrt{3}-\sqrt{2})$

$=11.442114$ in. $h_3=\frac{a}{b-b_1}\left[\sqrt{\frac{(n-3)b_1^2+3b^2}{n}}-\sqrt{\frac{(n-2)b_1^2+2b^2}{n}}\right]$

$=36[2-\sqrt{3}]=9.6462$ in.

FIG. 63.

II.
1. 4 in.=the width DC of the small end,
2. 8 in.=the width AB of the large end, and
3. 36 in.=the length BC of the board.
4. ∴ 216 sq. in.$=\frac{1}{2}(AB+DC)\times BC$ $=\frac{1}{2}(8+4)\times 36=$the area of the board.
5. $\frac{1}{3}$ of 216 sq. in.=72 sq. in.=the area of each piece.
6. $AK=AB-KB(=DC)=8$ in.—4 in. =4 in. In the similar triangles AKD and DCE,
7. AK:DK::AB:BE, or 4 in.:36 in.::8 in.:BE. Whence,
8. $BE=(36\times 8)\div 4=72$ in. [triangle ABE.
9. ∴ $\frac{1}{2}(AB\times BE)=\frac{1}{2}(8\times 72)=288$ sq. in.=the area of the
10. $ABE-ABCD=288$ sq. in.—216 sq. in.=72 sq. in. =area of the triangle DCE.

11. $DCE+DCGF=72$ sq. in.$+72$ sq. in. $=144$ sq. in. $=$the area of the triangle FGE.
12. $DEC+DCGF+FGIH=72$ sq. in.$+72$ sq. in.$+72$ sq. in.$=216$ sq. in.$=$the area of the triangle HIE.
13. $FEG:DEC::EG^2:EC^2$, or
144 sq. in.$:72$ sq. in.$::GE^2:36^2$. Whence,
14. $GE=\sqrt{(144\times36^2)\div72}=36\sqrt{2}=50.911686$ inches.
15. $\therefore$ $GC=GE-CE=50.911686$ in.-36 in.$=14.911686$ in., the length of $FGCD$. Again,
16. $DEC:HIE::EC^2:EI^2$, or
72 sq. in.$:216$ sq. in.$::36^2:EI^2$. Whence,
17. $EI=\sqrt{(216\times36^2)\div72}=36\times\sqrt{3}=62.3538$ in.
18. $\therefore$ $GI=EI-EG=36\sqrt{3}-36\sqrt{2}=36(\sqrt{3}-\sqrt{2})$ $=11.442114$ in., the length of $HIGF$, and
19. $BI=EB=EI=72-36\sqrt{3}=36(2-\sqrt{3})=9.6462$ in., the length of $ABIH$.

III. $\therefore\begin{cases} BI=9.6462 \text{ in.,} \\ GI=11.442114 \text{ in., and} \\ GC=14.911686 \text{ in.} \end{cases}$

Prob. CLXII. To find the edge of the largest cube that can be cut from a sphere.

Formula.—$e=\sqrt{\dfrac{D^2}{3}}=\frac{1}{3}\sqrt{3}D=.57735\times D$, where D is the diameter of the sphere.

Rule.—*Divide the square of the diameter of the sphere by three and extract the square root of the quotient; or, multiply the diameter by .57735.*

I. What is the edge of the largest cube that can be cut from a sphere 6 inches in diameter?

By formula, $e=\sqrt{\dfrac{D^2}{3}}=\sqrt{\frac{36}{3}}=6\times\sqrt{\frac{1}{3}}=\frac{1}{3}\sqrt{3}\times6=.57735\times6$ $=4.4641$ in.

II. $\begin{cases} 1. \text{ 6 in.=the diameter of the sphere.} \\ 2. \therefore .57735\times6 \text{ in.}=3.4641 \text{ in.=the edge of the largest cube that can be cut from the sphere.} \end{cases}$

III. $\therefore$ The edge of the largest cube that can be cut from a sphere whose diameter is 6 inches, is 3.4641 in.

Prob. CLXIII. To find the edge of the largest cube that can be cut from a hemisphere.

Formula.— $e=\sqrt{\dfrac{D^2}{6}}=\frac{1}{6}\sqrt{6}\times D=.408248\times D$.

Rule.—*Divide the square of the diameter by 6, and extract the square root of the quotient; or, multiply the diameter by .408248.*

I. What is the edge of the largest cube that can be cut from a hemisphere, the diameter of whose base is 12 inches?

By formula, $e=\sqrt{D^2\div6}=\sqrt{\frac{144}{6}}=12\sqrt{\frac{1}{6}}=\frac{1}{6}\sqrt{6}\times12=.408248\times12=4.899176$ in.

II. $\begin{cases} \text{1. 12 in.=the diameter of the base of the hemisphere.} \\ \text{2. } \therefore .408248\times12 \text{ in.}=4.899176 \text{ in.} \end{cases}$

III. ∴ The edge of the largest cube that can be cut from a hemisphere, the diameter of whose base is 12 feet, is 4.899176 in.

Prob. CLXIV. To find the diameter or radius of the three largest equal circles that can be inscribed in a circle of a given diameter or radius.

Formula.—$d=D\div(1+\frac{2}{3}\sqrt{3})=D\div2.1557=.4641\times D$ or $r=R\div(1+\frac{2}{3}\sqrt{3})=.4641\times R$.

Rule.—*Divide the diameter or radius of the given circle by 2.1557 and the quotient will be the diameter or radius of the three largest equal circles inscribed in it; or, multiply the diameter or radius by .4641, and the result will be the diameter or radius respectively of the required circles.*

I. A circular lot 15 rods in diameter is to have three circular grass beds just touching each other and the larger boundary; what must be the distance between their centers, and how much ground is left in the triangular space about the center?

By formula, $2r=2R\div(1+\frac{2}{3}\sqrt{3})=2R\div2.1557=\frac{15}{2.1557}=6.9615242$ rd.=the distance between their centers.

Construction.—Let AHE be the circular lot, C the center, and ACE any diameter. With E as a center and radius equal to CE describe an arc intersecting the circumference of the lot in H. Draw a tangent to the lot at E and produce the radius CH to intersect the tangent at B. Bisect the angle CBE and draw the bisector GB. It will meet the radius CE in G, the center of one of the grass beds. Draw GF perpendicular to CB. Then $GF=GE$, the radius of one of the grass beds. Draw EH. Then $EH=CH=EC$, and $CH=HB$, because the triangle EHB is isosceles.

A.
1. $CE=7\frac{1}{2}$ rd.$=R$, the radius of the lot.
2. $CB=2CH=2R$.
3. $EB=\sqrt{CB^2-CE^2}=\sqrt{(2R)^2-R^2}=R\sqrt{3}$. In the similar triangles CFG and CBE,
4. $CF:FG::CE::EB$, or $CF:GF::R:R\sqrt{3}$. But
5. $CF=CB-FB(=EB)=2R-R\sqrt{3}=R(2-\sqrt{3})$.
6. $\therefore R(2-\sqrt{3}):GF::R:R\sqrt{3}$. Whence,
7. $GF=\dfrac{R\sqrt{3}}{2-\sqrt{3}}=R(2\sqrt{3}-3)=7\frac{1}{2}(2\sqrt{3}-3)=7\frac{1}{2}\times .4641=3.48075$ rd.=the radius.

8. $GK=2r=2R(2\sqrt{3}-3)=6.9615$ rd., the distance between their centers.

II. B.

1. $GD=\sqrt{\overline{GK}^2-\overline{DK}^2}=\sqrt{4r^2-r^2}=r\sqrt{3}$.
2. $\frac{1}{2}(IK\times GD)=\frac{1}{2}(2r\times r\sqrt{3})=r^2\sqrt{3}=$the area of the triangle IGK.
3. Area $DKF=\frac{1}{6}$ of the small circle, because the angle DKF is 60°, or $\frac{1}{6}$ of 360°.
4. $\therefore$ Area $DKF=\frac{1}{6}\pi r^2$.
5. $\frac{1}{2}\pi r^2=3$ times $\frac{1}{6}\pi r^2$ $=$the area of the three parts of the small circles within the triangle IGK.
6. $\therefore r^2\sqrt{3}-\frac{1}{2}\pi r^2=r^2(\sqrt{3}-\frac{1}{2}\pi)=.16125368r^2$ $=.16125368\times[R(2\sqrt{3}-3)]^2=.16125368\times(21-12\sqrt{3})R^2=.16125368\times .2153904\times R^2$ $=.03473265\times R^2=.03473265\times(7\frac{1}{2})^2=1.953712$ sq. rd.$=$the area of the space inclosed.

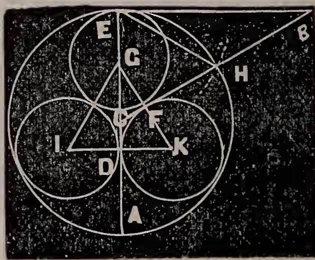


FIG. 64

III. $\therefore$ 6.9615 rd.$=$the distance between their centers, and 1.953712 sq. rd.$=$the area inclosed about the center of the given lot. *(R. H.A. p. 407, prob. 100.)*

Prob. CLXV. Having given the area inclosed by three equal circles to find the radius of a circle that will just inclose the three equal circles.

$$\textit{Formula.}—R=\sqrt{\left(\frac{A}{(2\sqrt{3}-3)^2(\sqrt{3}-\frac{1}{2}\pi)}\right)}$$
$$=\sqrt{\left(\frac{A}{.03473265}\right)}\text{, where } A \text{ is the area inclosed.}$$

Rule.—*Divide the area inclosed by .03473265 and extract the square root of the quotient, and the result will be the radius of the required circle.*

Prob. CLXVI. Having given the radius *a*, *b*, *c*, of the three circles tangent to each other, to find the radius of a circle tangent to the three circles.

$$\textit{Formula.}—r \text{ or } r'=\frac{abc}{2\sqrt{[abc(a+b+c)]}\mp(ab+ac+bc)},$$
the minus sign giving the radius of a tangent circle circumscribing the three given circles and the plus sign giving the radius of a tangent circle inclosed by the three given circles.

NOTE.—This formula is due to Prof. E. B. Seitz, Late Professor of Mathematics in the North Missouri State Normal School, Kirksville, Mo., of whom we give a biographical sketch accompanied by his photograph.

This formula is taken from the *School Visitor*, *Vol. II. p. 117*, with the

slight change that the plus sign is introduced for the case in which the tangent circle is inclosed by the three given circles. The problem of finding two circles tangent to three mutually tangent circles, is one supposed to have been proposed by Archimedes more than 2000 years ago, though the problem he proposed was not so general—the diameter of one of the given circles being equal to the sum of the diameters of the other two

The problem of finding *all* circles that can be drawn within three mutually tangent circles and tangent to each of them, has been simply and elegantly solved by D. H. Davison, Minonk, Ill. The above formula led him to his wonderful solution. For a complete and elegant solution, where he has actually computed and constructed 81 circles tangent to three given circles. see *School Visitor, Vol. VI., p. 80.*

Prob. CLXVII. To find the surface common to two equal circular cylinders whose axes intersect at right angles.

Formula.—$S=16R^2$, where R is the radius of the cylinders.

Rule.—*Multiply the square of the radius of the intersecting cylinders by 16.*

I. If the radius of two equal circular cylinders, intersecting at right angles is 4 feet, what is the surface common to both?

By formula, $S=16R^2=16\times4^2=256$ sq. ft.

II. 1. 4 ft. = the radius of the cylinders.
2. 16 sq. ft. $=4^2=$ the square of the radius of the cylinders
3. ∴ 16×16 sq. ft. = 256 sq. ft. = the surface common to the two cylinders.

III. ∴ 256 sq. ft. = the surface common to the two cylinders.

Prob. CLXVIII. To find the volume common to two equal circular cylinders whose axes intersect at right angles.

Formula.—$V=\frac{16}{3}R^3$, where R is the radius of the cylinder.

Rule.—*Multiply the cube of the radius of the cylinders by* $\frac{16}{3}$.

I. A man digging a well 3 feet in diameter, came to a log 3 feet in diameter lying directly across the entire well; what was the volume of the part of the log removed?

By formula, $V=\frac{16}{3}R^3=\frac{16}{3}(\frac{3}{2})^3=18$ cu. ft.

II. 1. 3 ft. = the diameter of the log and the well.
2. $1\frac{1}{2}$ ft. = the radius.
3. $3\frac{3}{8}$ cu. ft. $=(1\frac{1}{2})^3=$ the cube of the radius.
4. ∴ $\frac{16}{3}\times3\frac{3}{8}$ cu. ft. = 18 cu. ft., the volume of the part of the log removed.

III. ∴ The volume of the part of the log removed is 18 cu. ft.

Prob. CLXIX. To find the height of an object on the earth's surface by knowing its distance, the top of the object being visible above the horizon.

Let $BF=a$ be any object, $AB=t$ a tangent to the earth's surface from the top of the object, and $FE=D$ the diameter of the earth. Then by Geometry, $AB^2=BF(BF+FE)$, or $t^2=a(a+D)$. $\therefore a=\frac{t^2}{a+D}$. But a is very small as compared with the diameter of the earth and $AB=AF$ without appreciable error.

$\therefore$ ***Formula.***—$a=\frac{AF^2}{D}=\frac{c^2}{D}$, where c is the distance to the object from the point of observation.

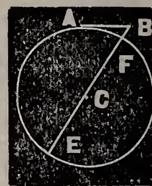


FIG. 65.

When $c=1$ mile, $a=\frac{1^2}{7912}=\frac{2}{3}$ ft., nearly.

Rule.—*Multiply the square of the distance in miles by $\frac{2}{3}$, and the result will be the height of the object in feet.*

I. What is the height of a steeple whose top can be seen at a distance of 10 miles?

By formula, $a=\frac{c^2}{D}=\frac{10^2}{7912}=\frac{10^2}{7912}\times5280=\frac{2}{3}\times10^2=66\frac{2}{3}$ ft.

II. $\begin{cases} 1. & \text{10 miles=the distance to the steeple.} \\ 2. & 100=10^2\text{=the square of the distance.} \\ 3. & \therefore \frac{2}{3}\text{ of }100=66\frac{2}{3}\text{ ft.=the height of the steeple.}\end{cases}$

III. $\therefore$ The height of the steeple is $66\frac{2}{3}$ ft.

Prob. CLXX. To find the distance to an object by knowing its height, the top only of the object being visible above the horizon.

$$\textit{Formula.}\text{—}c=\sqrt{aD}=\sqrt{\frac{aD}{5280}}=\sqrt{a\frac{7912}{5280}}=\sqrt{\tfrac{3}{2}a}.$$

Rule.—*Multiply the height of the object in feet by $\frac{3}{2}$ and extract the square root of the product, and the result will be the distance in miles.*

I. At what distance at sea can Mt. Aconcagua be seen, if its height is known to be 24000 feet?

By formula, $c=\sqrt{\frac{3}{2}a}=\sqrt{\frac{3}{2}\times24000}=\sqrt{36000}=190$ mi., nearly.

II. $\begin{cases} 1. & \text{24000 ft.=the height of the mountain} \\ 2. & \frac{3}{2}\times24000=36000. \\ 3. & \therefore \sqrt{36000}=10\sqrt{360}=190\text{ mi., nearly.}\end{cases}$

III. $\therefore$ Mt. Aconcagua can be seen at a distance of 190 miles.

Prob. CLXXI. Given the sum of the hypotenuse and perpendicular, and the base, to find the perpendicular.

Formula.—$p=\frac{s^2-b^2}{2s}$, where s is the sum of the hypotenuse and perpendicular, and b the base.

Rule.—*1. From the square of the sum of the hypotenuse and perpendicular subtract the square of the base, and divide the difference by twice the sum of the hypotenuse and perpendicular.*

2. To find the hypotenuse: To the square of the sum of the hypotenuse and perpendicular, add the square of the base and divide this sum by twice the sum of the hypotenuse and perpendicular.

I. A tree 120 feet high is broken off but not severed. The top strikes the ground 34 feet from the foot of the tree; what is the height of the stump?

By formula, $p=\frac{s^2-b^2}{2s}=\frac{120^2-34^2}{2\times120}=55\frac{11}{60}$ ft., the height of the stump.

II.
1. 120 ft.=the sum of the hypotenuse and perpendicular.
2. 34 ft.=the base, or the distance the top strikes from the foot of the tree.
3. 14400 sq. ft.$=120^2$=the square of said sum,
4. 1156 sq. ft.$=34^2$=the square of the base, and
5. 14400 sq. ft.—1156 sq. ft.=13244 sq. ft.=the difference.
6. $\therefore$ $13244\div(2\times120)=55\frac{11}{60}$ ft.=the height of the stump.

III. $\therefore$ The height of the stump is $55\frac{11}{60}$ feet.

NOTE.—This rule is easily derived from an algebraic solution. Thus: Let x=the perpendicular, $s-x$=the hypotenuse, and b=the base. Then, $x^2+b^2=(s-x)^2$, or $x^2+b^2=s^2-2sx+x^2$, and $x=\frac{s^2-b^2}{2s}$.

Prob. CLXXII. To find at what distance from the large end of the frustum of a right pyramid, a plane must be passed parallel to the base so that the two parts shall have equal solidities.

Formula.—$h=\frac{3V}{2(B+\sqrt{BB_2}+B^2)}$, where V is the volume of the frustum, B the area of the lower base, B_2 the area of the "dividing base," and $\sqrt{BB_2}$ the area of the mean base between the "dividing base" and and lower base.

Rule.—*1. Find the volume of the frustum by Prob. XCIII.*

2. Find the dimensions of the "dividing base" by extracting the cube root of half the sum of the cubes of the homologous dimensions of the upper and lower bases. Then find the area of the "dividing base."

3. Divide half the volume of the frustum by one-third of the sum of the areas of the lower base, "dividing base," and mean base between them, and the quotient will be the length of the lower part.

I. How far from the large end must a stick of timber, 20 feet long, 5 inches square at one end and 10 inches square at the other, be sawed in two parts, to have equal solidities?

By formula, $h=\dfrac{3V}{2(B+\sqrt{(BB_2)}+B_2)}=\dfrac{3\times\frac{1}{3}a(b^2+bc+c^2)}{2\left[b^2+b\times\sqrt[3]{\left(\frac{b^3+c^3}{2}\right)}+\left(\frac{b^3+c^3}{2}\right)^{\frac{2}{3}}\right]}=\dfrac{240(10^2+10\times5+5^2)}{2\left[10^2+10\times\sqrt[3]{\left(\frac{10^3+5^3}{2}\right)}+\sqrt[3]{\left(\frac{10^3+5^3}{2}\right)^2}\right]}$

$$=\frac{42000}{2(100+25\sqrt[3]{36}+\frac{75}{2}\sqrt[3]{6})}=\frac{1680}{8+2\sqrt[3]{36}+\sqrt[3]{6}}$$

$$=\frac{1680}{8+6.603855+5.4513618}=\frac{1680}{20.0552168}=83.76883+\text{in.}$$

Construction.—Let $ABCD—E$ be the piece of timber, $ABCD$ the lower base, $EFGH$ the upper base, and OL the altitude. Prolong the edges AH, BE, CF, and DG and the altitude OL till they meet in P. Draw KL to the middle point of AD, OI to the middle point of GH and draw PIK. Let $SMNR$ be the dividing base.

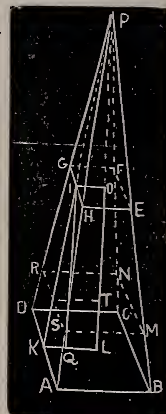


FIG. 66

II.
1. $AB=10$ in. $=b$, the side of the lower base.
2. $HE=5$ in. $=c$, the side of the upper base, and
3. $OL=20$ ft. $=240$ in. $=a$, the altitude.
4. $KQ=KL-QL(=IO)=\frac{1}{2}(b-c)=\frac{1}{2}(10$ in.—5 in$)=2\frac{1}{2}$ in. By similar triangles,
5. $KQ:QI::KL:PL$, or $\frac{1}{2}(b-c):a::\frac{1}{2}b:PL$. Whence,
6. $PL=\dfrac{ab}{b-c}=40$ ft.
7. $\therefore PO=PL-OL=\dfrac{ab}{b-c}-a=\dfrac{ac}{b-c}$ $=20$ ft.
8. $v=\frac{1}{3}PO\times HE^2=\frac{1}{3}ac^2=\frac{1}{3}\times240\times5^2$ $=2000$ cu. in., the volume of the pyramid $HEFG$.
9. $V=\frac{1}{3}OL\times(AB^2+AB\times HE+HE^2)=\frac{1}{3}a(b^2+bc+c^2)=14000$ cu. in., the volume of the frustum $ABCD—E$.
10. $\therefore \frac{1}{2}V=\frac{1}{2}$ of 14000 cu. in. $=7000$ cu. in., the volume of each part.
11. $v+\frac{1}{2}V=2000$ cu. in. $+7000$ cu. in. $=9000$ cu. in., the volume of the pyramid, $SMNR—P$, and
12. $v+V=2000$ cu. in. $+14000$ cu. in. $=16000$ cu. in., the volume of the pyramid $ABCD—P$. By the principle of similar solids, $[AB^3$, or
13. $HEFG\text{-}P:SMNR\text{-}P:ABCD\text{-}P::HE^3:SM^3:$
14. $v:v+\frac{1}{2}V:v+V::c^3:SM^3:b^3$. But

15. $v+\frac{1}{2}V=\frac{1}{2}[v+(v+V)]$, i. e., $v+\frac{1}{2}V$, or $SMNRP$ is an arithmetical mean between v and $v+V$, or $HEFG-P$ and $ABCD-P$.
17. $\therefore$ $SM^3=\frac{1}{2}(c^3+b^3)$, i. e., SM^3 is an arithmetical mean between HE^3 and AB^3, or c^3 and b^3. Whence,
18. $SM=\sqrt[3]{[\frac{1}{2}(c^3+b^3)]}=\sqrt[3]{[\frac{1}{2}(5^3+10^3)]}=\frac{5}{2}\sqrt[3]{36}=$ 8.2548188+in.
19. $SM^2=\sqrt[3]{[\frac{1}{2}(c^2+b^2)]^2}=(\frac{5}{2}\sqrt[3]{36})^2=\frac{75}{2}\sqrt[3]{6}=68.14202$ sq. in=the area of the dividing base.
20 $\sqrt{(SM^2\times AB^2)}=SM\times AB=\frac{5}{2}\sqrt[3]{36}\times10=25\sqrt[3]{36}=$ 82.54818 sq. in.=the area of the mean base of the part cut from the frustum.
21. $\therefore$ $\frac{1}{3}LT(AB^2+SM\times AB+SM^2)=\frac{1}{3}LT(b^2+\sqrt[3]{[\frac{1}{2}(b^3+c^3)]}\times b+\sqrt[3]{[\frac{1}{2}(b^3+c^3)]^2})=\frac{1}{3}LT[10^2+\frac{5}{2}\sqrt[3]{(36)}\times10+(\frac{5}{2}\sqrt[3]{36})^2]=\frac{1}{3}LT(100+82.54818+68.14202)=\frac{1}{3}LT\times250.6902=LT\times83.5634=$ the volume of the frustum $ABCD-M$ .But $\Gamma-M$.
23. $\frac{1}{2}V=7000$ cu. in.=the volume of the frustum $ABCD$
24. $\therefore$ $LT\times83.5634=7000$ cu. in. Whence,
25. $LT=7000\div83.5634=83.76883$ in.=6 ft. 11.76883 in., the length.

III. $\therefore$ The stick must be cut in two at a distance of 83.76883 in., or 6 ft. 11.76883 in., from the large end.

NOTE.—The frustum of a cone may be divided into two equal parts in the same manner. The frustum of a pyramid or a cone can be divided into any number of equal parts on the same principle as that for dividing a trapezoid into n equal parts, Prob. CLXI.

I. The area of a rectangle whose length is 20 rods is 120 sq. rods; what is the area of a similar rectangle whose length is 30 rods?

Principle.—*Similar areas are to each other as the squares of their like dimentions or as the squares of any other homologous lines.*

II.
1. 30 rods=the length of the given rectangle, and
2. 120 sq. rd.=its area.
3. 30 rods=the length of the required rectangle.
4. $\therefore$ $20^2:30^2::120$ sq. rd. : (?). Whence,
5. $?=(120\times30^2)\div20^2=270$ sq. rd.

III. $\therefore$ The area of the rectangle is 270 sq. rd.

I. The area of a rectangle whose width is 7 feet, is 210 sq. ft.; what is the length of a similar rectangle whose area is 2100 sq. ft.

II.
1. 210 sq. ft.=the area of given rectangle, and
2. 7 ft.=its width. Then
3. $210\div7=30$ ft.=its length.
4. $\therefore$ 210 sq. ft.: 2100 sq. ft. : : 30^2 :(?). Whence,
5. $?=(2100\times30^2)\div210=300$ ft.=the length of the required rectangle.

III. The length of the required rectangle is 300 feet.

I. If the weight of a well proportioned man, 5 feet in height, be 125 lbs., what will be the weight of a similarly proportioned man 6 feet high?

Principle.—*Similar solids are to each other as the cubes of their like dimensions or as the cubes of any other homologous lines.*

II.
1. 5 ft.=the height of the first man, and
2. 125 lbs.=his weight.
3. 6 ft.=the height of the second man.
4. $\therefore$ $5^3 : 6^3 :: 125$ lbs : (?). Whence,
5. $? = (125 \times 6^3) \div 5^3 = 216$ lbs., the weight of the second man.

III. $\therefore$The weight of the man whose height is 6 feet, is 216 lbs.

I. James Page has a circular garden 10 rods in diameter. How many trees can be set in it so that no two shall be within 10 feet of each other and no tree within $2\frac{1}{2}$ feet of the fence?

Construction.—Let ABC be the circular garden, AC it diameter, and O its, center. Then with O as a center and radius $AO = \frac{1}{2}$of $(10 \times 16\frac{1}{2}$ ft.$-2 \times 2\frac{1}{2}$ ft.), or 80 ft, discribe the circle *abcdef*, and in it describe the regular hexagon *abcdef.* Then $aO = ab = 80$ ft. Begin at the center of the circle and put 8 trees 10 ft. apart on each radii, aO, bO, cO, dO. eO, and fO. Then joining these points by lines drawn parallel to the diameter of the circle as shown in the figure, their points of intersection will mark the position of the trees. Hence, the trees are arranged in hexagonal form about the center. The first hexagonal row contains 6 trees, the second, 12, the third 18, and so on. Since the radius of the circle on which the trees are placed is 80 feet and the trees 10 feet apart, there will be 8 hexagonal rows.

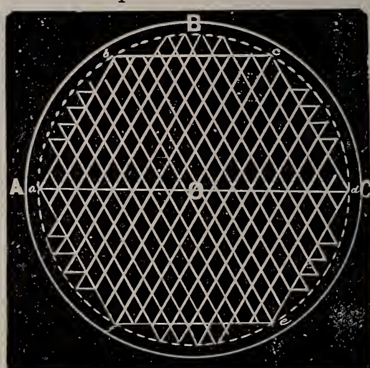


FIG. 67.

II.
1. 6=the number of trees in the first hexagonal row.
2. 12=the number of trees in the second hexagonal row.
3. 48=the number of trees in the eighth hexagonal row.
4. $\therefore$ $216 = \frac{1}{2}(6+48) \times 8$=the number of trees in the eight hexagonal rows.
5. $24 = 6 \times 4$=the number of trees at the sides of the hexagon *abcdef.*
6. $\therefore$ $216+24+1$, the tree at the center,$=241$=the number of trees that can be set in the garden.

III. $\therefore$ There can be set in the garden, 241 trees.

(*Greenleaf's Nat'l Arith., p. 444, prob. 25.*)

I. There is a ball 12 feet in diameter on top of a pole 60 feet high. On the ball stands a man whose eye is 6 feet above the ball; how much ground beneath the ball is invisible to him?

Construction.—Let BE be the pole, L the center of ball, and A the position of the man's eye. Draw AFC tangent to the ball at F and draw LF and BC. Then the triangle AFL is right-angled at F.

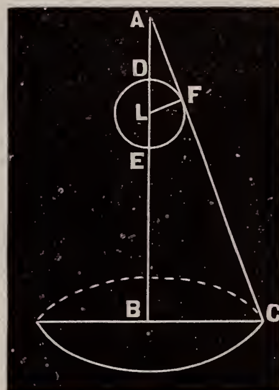


FIG. 68.

II.
1. 60 ft.$=BE$, the length of the pole.
2. 12 ft.$=ED$, the diameter of the ball, and
3. 6 ft.$=AD$, the height of the man's eye above the ball.
4. 12 ft.$=AD+DL=AL$. Now
5. $AF=\sqrt{(AL^2-LF^2)}$ $=\sqrt{(12^2-6^2)}=6\sqrt{3}$ ft. In the similar triangles ALF and ACB,
6. $AF:LF::AB:BC$, or $6\sqrt{3}$ ft. : 6 ft. : : (6 ft.+12 ft. +60 ft.), or 78 ft. : BC.
7. $\therefore BC=(6\times78)\div6\sqrt{3}=78\div\sqrt{3}=\frac{1}{3}\times78\sqrt{3}=26\sqrt{3}$ ft.
8. $\therefore \pi BC^2=\pi(26\sqrt{3})^2=6371.1498932$ sq. ft.=the area of the circle over which the man can not see.

III. $\therefore$ 6371.1498932 sq. ft.=the area of the invisible ground beneath the ball.

I. Three women own a ball of yarn 4 inches in diameter. How much of the diameter of the ball must each wind off, so that the may share equally?

II.
1. 4 in.=the diameter of the ball. Then
2. $\frac{1}{6}\pi(4)^3=\frac{32}{3}\pi$=the volume of the ball.
3. $\frac{1}{3}$ of $\frac{32}{3}\pi=\frac{32}{9}\pi$=each woman's share.
4. $\frac{32}{3}\pi-\frac{32}{9}\pi=\frac{64}{9}\pi$=the volume of the ball after the first has unwound her share. But
5. $\frac{1}{6}\pi D^3$=the volume of any sphere whose diameter is D.
6. $\therefore \frac{1}{6}\pi D^3=\frac{64}{9}\pi$. Whence,
7. $D^3=\frac{64}{9}\pi\div\frac{1}{6}\pi=\frac{128}{3}$, and
8. $D=\sqrt[3]{\frac{128}{3}}=4\sqrt[3]{\frac{2}{3}}=\frac{4}{3}\sqrt[3]{18}=\frac{4}{3}\times2.6207414=3.4943219$ in., diameter of the ball after the first unwound her share.
9. $\therefore$ 4 in.—3.4943219 in=.5056781 in., what the diameter was reduced by the first woman.
10. $\frac{64}{9}\pi-\frac{32}{9}\pi=\frac{32}{9}\pi$, the volume of the ball after the second had unwound her share.

11. $\therefore \sqrt[3]{(\frac{32}{9}\pi \div \frac{1}{6}\pi)} = 4\sqrt[3]{\frac{1}{3}} = \frac{4}{3}\sqrt[3]{9} = \frac{4}{3} \times 2.0800837$ $= 2.5734448$ in., the diameter of the ball after the second woman unwoud her share.
12. $\therefore$ 3.4943219 in.—2.5734448 in.=.7208771 in., what the diameter was reduced by the second woman.

III. $\therefore$ The diameter was diminished .5056781 in. by the first woman,
.7208771 in. by the second woman, and
2.7734448 in. by the third woman.

(*Milne's Prac. Arith., p. 335, prob. 8.*)

NOTE.—The following are the formulas to divide a sphere into n equal parts, the parts being concentric:

$$D-D_1=\sqrt[3]{\left(\frac{n-1}{n}\right)}D;\ D_1-D_2=\left[\sqrt[3]{\left(\frac{n-1}{n}\right)}-\sqrt[3]{\left(\frac{n-2}{n}\right)}\right]D;\ D_2-D_3=\left[\sqrt[3]{\left(\frac{n-2}{n}\right)}-\sqrt[3]{\left(\frac{n-3}{n}\right)}\right]D;$$

$$D_3-D_4=\left[\sqrt[3]{\left(\frac{n-3}{n}\right)}-\sqrt[3]{\left(\frac{n-4}{n}\right)}\right]D,$$ and so on, where D is the diameter of the sphere; D_1, the diameter after the first part is taken off; D_2, the diameter after the second part is taken off; and so on. Then $D-D_1$, D_1-D_2, &c, are portions of the diameter taken off by each part.

I. A park 20 rods square is surrounded by a drive which contains $\frac{19}{100}$ of the whole park; what is the width of the drive?

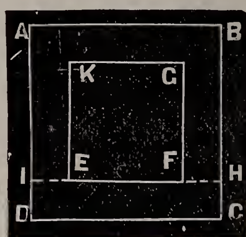


FIG. 69.

II.
1. 20 rd.$=AD=DC$, a side of the park.
2. 400 sq. rd.$=20^2=$the area of the park $ABCD$.
3. $\frac{19}{100}$ of 400 sq. rd.=76 sq. rd.=the area of the path.
4. 400 sq. rd.—76 sq. rd.=324 sq. rd. =the area of the square $EFGH$.
5. $EF=\sqrt{(324)}=18$ rd., the side of the square $EFGH$.
6. $\therefore IH-EF=20$ rd.—18rd.=2 rd., twice the width of the path.
7. $\therefore$ 1 rd.$=\frac{1}{2}$ of 2 rd.=the width of the path.

III. $\therefore$ The width of the path is 1 rod.

I. My lot contains 135 sq. rd., and the breadth is to the length as 3 to 5; what is the width of a road which shall extend from one corner half around the lot and occupy $\frac{1}{4}$ of the ground.

Construction.—Let $ABCD$ be the lot, and $DABSNR$ the road. Produce AB, till BE is equal to AD. Then AE is equal to $AB+AD$. On AE, construct the square $AEFG$, and

on EF and GF respectively, lay off EI and FK equal to AB. Then construct the rectangles $BEIH$, $ILKF$, and $KMDG$. They will each be equal to $ABCD$, for their lengths and widths are equal to the length and width of $ABCD$. Continue the road around the square. Then the area of the road around the square is four times the area of the road $DABSNR$.

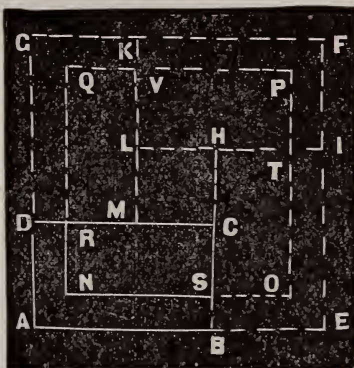


FIG. 70.

II.
1. $\frac{3}{3}$=the width AD of the lot. Then
2. $\frac{5}{3}$=the length AB.
3. $\frac{5}{3}\times\frac{3}{3}$=135 sq. rd., the area of the lot.
4. $\frac{1}{3}\times\frac{3}{3}=\frac{1}{5}$ of 135 sq. rd. =27 sq. rd., and
5. $\frac{3}{3}\times\frac{3}{3}=(\frac{3}{3})^2$=3 times 27 sq.rd.=81 sq. rd.
6. ∴ $\frac{3}{3}=\sqrt{81}$=9 rd., the width AD,
7. $\frac{1}{3}=\frac{1}{3}$ of 9 rd.=3 rd., and
8. $\frac{5}{3}$=5 times 3 rd.=15 rd., the length AB.
9. 15 rd+9 rd.=24 rd.= AE, the side of the square $AEFG$.
10. ∴ 576 sq. rd.=24^2= the area of the square $AEFG$.
11. $33\frac{3}{4}$ sq. rd.=$\frac{1}{4}$ of 135 sq. rd.=the area of the road $DABSNR$.
12. ∴ 135 sq. rd.=4×$33\frac{3}{4}$ sq. rd.=the area of the road around the square. Then
13. 576 sq. rd.—135 sq. rd.=441 sq. rd., the area of the square $NOPQ$.
14. ∴ 21rd.=$\sqrt{441}$=NO, a side of the square $NOPQ$.
15. AE—NO=24 rd.—21 rd.=3rd.=twice the width of the road.
16. ∴ $1\frac{1}{2}$ rd.=$24\frac{3}{4}$ ft.=$\frac{1}{2}$ of 3 rd.=the width of the road.

III. ∴ The width of the road is $24\frac{3}{4}$ ft.

(*R. H. A., p.407, prob. 99.*)

I. The length and breadth of a ceiling are as 6 to 5; if each dimension were one foot longer, the area would be 304 sq. ft.; what are the dimensions?

Construction.—Let $ABCD$ be the ceiling, AB its width and BC its length. Let $AIGE$ be the ceiling when each dimension is increased one foot. On BC, lay off BK equal to AB and draw LK parallel to AB. Then $ABKL$ is a square whose side is the width of the ceiling.

II.

1. $\frac{5}{5}=AB$, the width of the ceiling. Then
2. $\frac{6}{5}=BC$, the length, and
3. $\frac{6}{5}\times\frac{5}{5}=AB\times BC=$the area of the ceiling.
4. $\frac{6}{5}\times1=BC\times BI$, BI being 1 foot,=the area of the rectangle $BCHI$.
5. $\frac{5}{5}\times1=DC\times CF$, CF being 1 foot,=the area of the rectangle $DCFE$.
6. 1 sq. ft.$=1^2=$the area of the square $CFGH$.
7. $\therefore\frac{6}{5}\times\frac{5}{5}+\frac{6}{5}\times1+\frac{5}{5}\times1+1$ sq. ft. =the area of $AIGE$. But

FIG. 71.

8. $\frac{6}{5}\times1+\frac{5}{5}\times1=\frac{11}{5}\times1=$rectangle $BIHC+$rectangle $DCFE$, i. e., it equals a rectangle whose length is $\frac{6}{5}+\frac{5}{5}$, or $\frac{11}{5}$, and width 1 ft.
9. $\therefore\ \frac{6}{5}\times\frac{5}{5}+\frac{6}{5}\times1+\frac{5}{5}\times1+1$ sq. ft.$=\frac{6}{5}\times\frac{5}{5}+\frac{11}{5}\times1+1$ sq. ft. =the area of $AIGE$. But
10. 304 sq. ft.=the area of $AIGE$.
11. $\therefore\ \frac{6}{5}\times\frac{5}{5}+\frac{11}{5}\times1+1$ sq. ft.=304 sq. ft. Whence,
12. $\frac{6}{5}\times\frac{5}{5}+\frac{11}{5}\times1=303$ sq. ft=the area of $AIHCFE$. But
13. $\frac{11}{5}=\frac{11}{5}\times\frac{5}{5}$, in which $\frac{11}{5}$ is $\frac{11}{5}$ ft.; for a rectangle whose length is $\frac{11}{5}$, and the width 1 ft., has the same area as a rectangle whose width is $\frac{11}{5}$ ft. and length 1, or $\frac{5}{5}$.
14. $\therefore\ \frac{6}{5}\times\frac{5}{5}+\frac{11}{5}\times\frac{5}{5}=303$ sq. ft., in which $\frac{11}{5}$ is $\frac{11}{5}$ ft.
15. $\frac{1}{5}\times\frac{5}{5}+\frac{11}{30}\times\frac{5}{5}=50\frac{1}{2}$ sq. ft.$=\frac{1}{6}$ of $(\frac{6}{5}\times\frac{5}{5}+\frac{11}{5}\times\frac{5}{5})=\frac{1}{6}$ of 303 sq. ft.,
16. $\frac{5}{5}\times\frac{5}{5}+\frac{11}{6}\times\frac{5}{5}=252\frac{1}{2}$ sq. ft.$=5\times(\frac{1}{5}\times\frac{5}{5}+\frac{11}{30}\times\frac{5}{5})=5\times50\frac{1}{2}$ sq. ft. But
17. $\frac{5}{5}\times\frac{5}{5}=$the area of the square $ABKL$, and
18. $\frac{11}{6}\times\frac{5}{5}=$the area of the rectangle $ALNP$ whose length AL is $\frac{5}{5}$ and width LN $\frac{11}{6}$ ft.
19. $\frac{1}{2}$ of$(\frac{11}{6}\times\frac{5}{5})=\frac{11}{12}\times\frac{5}{5}=$half the rectangle $ALNP=$ the rectangle $OMNP$, which put to the side AB of the square $ABKL$ as in the figure.
20. $\frac{5}{5}\times\frac{5}{5}+\frac{11}{6}\times\frac{5}{5}=252\frac{1}{2}$ sq. ft.=the area of $SRAOMK$.
21. $\frac{121}{144}$ sq. ft.$=(\frac{11}{12})^2=$the area of the square $RQOA$, since AR is $\frac{11}{12}$ ft.
22. $\therefore\ \frac{5}{5}\times\frac{5}{5}+\frac{11}{6}\times\frac{5}{5}+\frac{121}{144}$ sq. ft.$=252\frac{1}{2}$ sq. ft.$+\frac{121}{144}$ sq. ft. $=\frac{36481}{144}$ sq. ft.=the area of $(SRAOMK+RQOA)$. =the area of $SQMK$.
23. $\frac{5}{5}+\frac{11}{12}$ ft.$=\sqrt{\frac{36481}{144}}=\frac{191}{12}$ ft.=the side SK of the square $SQMK$.
24. $\frac{5}{5}=\frac{191}{12}$ ft.$-\frac{11}{12}$ ft.$=\frac{180}{12}$ ft.=15 ft.$=SK-SB=BK$ $=AD$, the width of the ceiling.

25. $\frac{1}{5}=\frac{1}{5}$ of 15 ft.=3 ft., and
26. $\frac{6}{5}=6$ times 3 ft.=18 ft.=BC, the length of the ceiling.

III. $\therefore$ $\begin{cases} \text{15 ft.=the width of the ceiling, and} \\ \text{18 ft.=the length.} \end{cases}$

Remark.—In this solution there is but one algebraic operation; viz., extracting the square root of the binomial expression, $(\frac{5}{5}\times\frac{5}{5}+\frac{17}{6}\times\frac{5}{5}+\frac{121}{144}$ sq. in.), in step 23. This might have been omitted and then the solution would have been purely arithmetical; for, the area of the square $SQMK$ being known, as shown by step 22, its side SK could have been found by simply extracting the square root of its area, $\frac{36841}{144}$ sq. ft. Then by subtracting SB, which is $\frac{11}{12}$ ft., from SK, we would get $BK(=AB)$, the width of the ceiling.

The following solution is quite often given in the schoolroom:
$304\div(5\times6)=10+$. $\sqrt{10}=3+$.
$5\times3=15$, the width and $6\times3=18$, the length.

I. A tin vessel, having a circular mouth 9 inches in diameter, a bottom $4\frac{1}{2}$ inches in diameter, and a depth of 10 inches, is $\frac{1}{4}$ part full of water; what is the diameter of a ball which can be put in and just be covered by the water?

Construction.—Let $ABCD$ be a vertical section of the vessel, AB the top diameter, DC the bottom diameter, and EF the altitude. Produce AD, BC, and EF till they meet in G. Draw MC parallel to EF. In the triangle ACB inscribe the largest circle IEP and let Q be its center. Draw the radius IQ. Now

1. $AE=\frac{1}{2}AB=R=4\frac{1}{2}$ in.=the radius of the mouth.
2. $CF=\frac{1}{2}DC=r=2\frac{1}{4}$ in., the radius of the bottom, and
3. $EF=a=10$ in., the altitude of the vessel.
4. $MB=EB-EM(=FC)=R-r=4\frac{1}{2}$ in. $-2\frac{1}{4}$ in.$=2\frac{1}{4}$ in. In the similar triangles BMC and BEG,
5. $MB:MC::EB:EG$, or $R-r:a::R:EG$. Whence,
6. $EG=\dfrac{aR}{R-r}=\dfrac{10\times4\frac{1}{2}}{4\frac{1}{2}-2\frac{1}{4}}=20$ in., the altitude of the triangle AGB.
7. $IQ=\dfrac{2\triangle AGB}{AB+AG+BG}=\dfrac{AB\times EG}{AB+AG+BG}$ $=\dfrac{2Ra}{AB+BG+BG}$. But

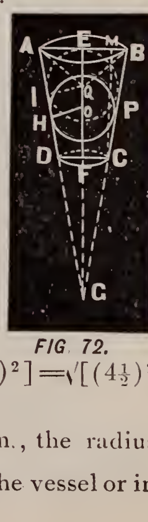


FIG. 72.

8. $AG=BG=\sqrt{(EB^2+EG^2)}=\sqrt{[R^2+(2a)^2]}=\sqrt{[(4\frac{1}{2})^2+20^2]}=\sqrt{(420\frac{1}{4})}=20\frac{1}{2}$ in.
9. $\therefore IQ=\dfrac{4Ra}{2R+2\sqrt{(R^2+a^2)}}=\dfrac{4\frac{1}{2}\times20}{4\frac{1}{2}+20\frac{1}{2}}=3\frac{3}{5}$ in., the radius of the largest sphere that can be put in the vessel or in

II.

the cone AGB.

10. $\frac{4}{3}\pi(IQ)^3=\frac{4}{3}\pi\left(\frac{R2a}{R+\sqrt{(R^2+4a^2)}}\right)^3=\frac{4}{3}\pi(3\frac{3}{5})^3=\frac{7776}{125}\pi$ =the volume of the largest sphere that can be put in the cone AGB.

11. $\frac{1}{3}EG\times\pi EB^2=\frac{1}{3}\pi 2aR^2=\frac{1}{3}\pi\times20\times(4\frac{1}{2})^2=135\pi$, the volume of the cone AGB.

12. $\therefore \frac{1}{3}\pi 2aR^2-\frac{4}{3}\pi\left(\frac{2aR}{R+\sqrt{(R^2+2a^2)}}\right)^3=\frac{1}{3}\pi 2aR^2\times\left[1-\frac{2a^2R}{[R+\sqrt{(R^2+2a^2)}]^3}\right]=135\pi-\frac{7776}{125}\pi=\frac{9099}{125}\pi=$ the quantity of water in the cone which will just cover the largest ball that can be put in the cone AGB.

13. $\frac{1}{3}\pi FG\times FC^2=\frac{1}{4}\pi ar^2=\frac{1}{3}\pi\times10\times(2\frac{1}{4})^2=\frac{135}{8}\pi$, the volume of the cone DGC.

14. $\therefore \frac{1}{3}\pi ar^2+\frac{1}{4}$ of the volume of the vessel$=\frac{135}{8}\pi+\frac{1}{4}$ of the volume of the vessel=the quantity of water in the cone necessary to cover the required ball. But

15. $\frac{1}{3}\pi a(R^2+Rr+r^2)=\frac{1}{3}\pi 10[(4\frac{1}{2})^2+4\frac{1}{2}\times2\frac{1}{4}+(2\frac{1}{4})^2]$ $\frac{945}{8}\pi$, the volume of the vessel, by Prob. XCIII.

16. $\therefore \frac{1}{3}\pi ar^2+\frac{1}{4}$ of the volume of the vessel$=\frac{1}{3}\pi ar^2+\frac{1}{4}$ of $\frac{1}{3}\pi a(R^2+Rr+r^2)=\frac{1}{3}\pi a[r^2+\frac{1}{4}(R^2+Rr+r^2)]=\frac{135}{8}\pi+\frac{1}{4}$ of $\frac{945}{8}\pi=\frac{1485}{32}\pi$, the quantity necessary to cover the required ball.

17. $\therefore$ The quantity of water necessary to cover the largest ball: the quantity of water necessary to cover the required ball : : (radius)^3 of largest ball : (radius)^3 of required ball. Hence,

18. $\frac{1}{3}\pi 2aR^2\left[1-\frac{4a^2R}{[R+\sqrt{(R^2+4a^2)}]^3}\right]:\frac{1}{3}\pi a[r^2+\frac{1}{4}(R^2+Rr+r^2)]::\left(\frac{R2a}{R+\sqrt{(R^2+4a^2)}}\right)^3:HO^3$, or

19. $\frac{9099}{125}\pi:\frac{1485}{32}\pi::(3\frac{3}{5})^3:HO^3$. Whence,

20. $\frac{3}{5}\sqrt[3]{(337)}:\frac{3}{2}\sqrt[3]{(\frac{55}{2})}::3\frac{3}{5}:HO$. Whence,

21. $HO=\sqrt[3]{\left(\frac{R[r^2+\frac{1}{4}(R^2+Rr+r^2)]}{1-\frac{2a^2R}{[R+\sqrt{(R^2+4a^2)}]^3}}\right)}=[\frac{3}{2}\sqrt[3]{(\frac{55}{2})}\times3\frac{3}{5}]$ $\div\frac{3}{5}\sqrt[3]{337}=9\sqrt[3]{(\frac{55}{674})}$, and

22. $18\sqrt[3]{(\frac{55}{674})}=6.1967+$in., the diameter of the required ball.

III. $\therefore$ The diameter of the required ball is 6.1967+ in.

I. I have a garden in the form of an equilateral triangle whose sides are 200 feet. At each corner stands a tower; the height of the first tower is 30 feet, the second 40 feet, and the third 50 feet. At what distance from the base of each tower

must a ladder be placed, so that without moving it at the base it may just reach the top of each, and what is the length of the ladder?

Construction.—Let ABC be the triangular garden and AD, BE, and CF the towers at the corners. Connect the tops of the towers by the lines ED and DF. From G and H, the middle points of DE and DF, draw GM and HN perpendicular to DE and DF, and at M and N draw perpendiculars to AB and AC in the triangle ABC, meeting at O. Then O is equally distant from D and E. For, since M is equally distant from D and E, and MO perpendicular to the plane $ABED$, every point of MO is equally distant from D and E. For a like reason, every point of NO is equally distant from D and F; hence, O their point of intersection, is equally distant from D, E, and F and is, therefore, the point where the ladder must be placed. Draw DI and DJ parallel to AB and AC, GK and HL perpendicular to AB and AC, MP perpendicular to AC, and OR parallel to NP. Draw the lines OB, OC, and OA, the required distances from the base of the ladder to the bases of the towers. Draw EO, the length of the ladder.

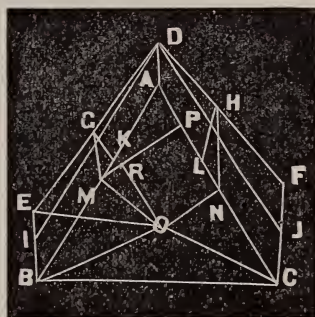


FIG. 73.

1. $AB=BC=AC=200$ ft.$=s$, the side of the triangle.
2. $FC=50$ ft.$=a$, the height of the first tower,
3. $EB=40$ ft.$=b$, the height of the second tower, and
4. $AD=30$ ft.$=c$, the height of the third tower. Let
5. $h=\sqrt{[AB^2-(\frac{1}{2}AC)^2]}=\sqrt{[s^2-(\frac{1}{2}s)^2]}=\frac{1}{2}\sqrt{3}s=100\sqrt{3}$ ft$=$the perpendicular from B to the side AC.
6. $EI=BE-BI(=AD)=(b-c)=40$ ft.-30 ft. $=10$ ft.
7. $GK=\frac{1}{2}(EB+AD)=\frac{1}{2}(b+c)=\frac{1}{2}(40 \text{ ft.}+30 \text{ ft.})=35$ ft. In the similar triangles DIE and GKM,
8. $DI:IE::GK:KM$, or $s:b-c::\frac{1}{2}(b+c):KM$.
9. $\therefore KM=\dfrac{b^2-c^2}{2s}=\dfrac{40^2-30^2}{2\times200}=1\frac{3}{4}$ ft.,
10. $AM=AK+KM=\frac{1}{2}s+\dfrac{b^2-c^2}{2s}=\dfrac{s^2+b^2-c^2}{2s}$ $=101\frac{3}{4}$ ft., and
11. $BM=AB-AM=s-\dfrac{s^2+b^2-c^2}{2s}=\dfrac{s^2+c^2-b^2}{2s}$ $=98\frac{1}{4}$ ft. In like manner,
12. $HL=\frac{1}{2}(a+c)=\frac{1}{2}(50 \text{ ft.}+30 \text{ ft.})=40$ ft.,

II. A.

13. $LN=\frac{a^2-c^2}{2s}=4$ ft.,

14. $AN=AL+LN=\frac{1}{2}s+\frac{a^2-c^2}{2s}=\frac{s^2+a^2-c^2}{2s}=104$ ft.

15. $NC=AC-AN=s-\frac{s^2+a^2-c^2}{2s}=\frac{s^2+c^2-a^2}{2s}=$ 96 ft. By similar triangles,

16. $AB:AL::AM:AP$, or $s:\frac{1}{2}s:(s^2+b^2-c^2)\div 2s:AP$. Whence,

17. $AP=(s^2+b^2-c^2)\div 4s=50\frac{7}{8}$ ft.

18. $\therefore PL=AL-AP=[\frac{1}{2}s-(s^2+b^2-c^2)\div 4s]=(s^2+c^2-b^2)\div 4s=49\frac{1}{8}$ ft.

19. $RO=PN=PL+LN=(s^2+c^2-b^2)\div 4s+(a^2-c^2)\div 2s=(s^2+2a^2-b^2-c^2)\div 4s=53\frac{1}{8}$ ft. By similar triangles,

20. $AB:BL::AM:MP$, or $s:\frac{1}{2}\sqrt{3}s::(s^2+b^2-c^2)\div 2s:MP$. Whence,

21. $MP=[(s^2+b^2-c^2)\div 4s]\times\sqrt{3}=50\frac{7}{8}\sqrt{3}$ ft. By similar triangles,

22. $MP:AP::RO:RM$, or $[(s^2+b^2-c^2)\div 4s]\sqrt{3}:(s^2+b^2-c^2)\div 4s::(s^2+2a^2-b^2-c^2)\div 4s:RM$.

23. $RM=(s^2+2a^2-b^2-c^2)4\sqrt{3}s=[(s^2+2a^2-b^2-c^2)\div 12s]\sqrt{3}=17\frac{17}{24}\sqrt{3}$ ft. Again

24. $MP:MA::RO:OM$, or $[(s^2+b^2-c^2)\div 4s]\sqrt{3}:(s^2+b^2-c^2)\div 2s::(s^2+2a^2-b^2-c^2)\div 4s:OM$.

25. $\therefore OM=(s^2+2a^2-b^2-c^2)\div 2\sqrt{3}s=[(s^2+2a^2-b^2-c^2)\div 6s]\sqrt{3}=35\frac{5}{12}\sqrt{3}$ ft.

26. $ON=RP=MP-RM=[(s^2+b^2-c^2)\div 4s]\sqrt{3}-(s^2+2a^2-b^2-c^2)\div 12s]\sqrt{3}=[(s^2-a^2+2b^2-c^2)\div 6s]\sqrt{3}=33\frac{1}{6}\sqrt{3}$ ft. Then

27. $OC=\sqrt{(ON^2+NC^2)}=\sqrt{\left[\left(\frac{s^2-a^2+2b^2-c^2}{6s}\sqrt{3}\right)^2+\left(\frac{s^2+c^2-a^2}{2s}\right)^2\right]}=\sqrt{[(33\frac{1}{6}\sqrt{3})^2+96^2]}=\sqrt{12516\frac{1}{12}}$ $=111.8796+$ft.

28. $OA=\sqrt{(ON^2+AN^2)}=\sqrt{\left[\left(\frac{s^2-a^2+2b^2-c^2}{6s}\sqrt{3}\right)^2+\left(\frac{s^2+c^2-a^2}{2s}\right)^2\right]}=\sqrt{[(33\frac{1}{6}\sqrt{3})^2+104^2]}$ $=\sqrt{14116\frac{1}{12}}=118.8111+$ft.

29. $OB=\sqrt{(OM^2+MB^2)}=\sqrt{\left[\left(\frac{s^2+2a^2-b^2-c^2}{6s}\sqrt{3}\right)^2+\left(\frac{s^2+c^2-b^2}{2s}\right)^2\right]}=\sqrt{[(35\frac{5}{12}\sqrt{3})^2+(98\frac{1}{4})^2]}$ $=\frac{1}{4}\sqrt{214657\frac{1}{3}}=115.8278+$ft.

B. 1. $OE=\sqrt{(BE^2+OB^2)}=\sqrt{[(\frac{1}{4}\sqrt{214657\frac{1}{3}})^2+40^2]}$, $=\sqrt{(13416\frac{1}{2}+1600)}=\sqrt{15016\frac{1}{12}}=122.5402+$ft.$=$the length of the ladder.

III. ∴ $\begin{cases} \text{1. } 111.8796+\text{ft.}=\text{the distance from base of the ladder to the base of the tower } FC, \\ \text{2. } 118.8111+\text{ft.}=\text{the distance from the base of the ladder to the base of the tower } AD. \\ \text{3. } 115.8278+\text{ft.}=\text{the distance from the base of the ladder to the bsse of the tower } BE, \text{ and} \\ \text{4. } 122.5402+\text{ft.}=\text{the length of the ladder.} \end{cases}$

(*Greenleaf's Nat'l Arith., p. 444, prob. 38.*)

Remark.—When the sides of the triangle are unequal, proceed in the same manner as above. In some cases the base of the ladder will fail without the triangle.

I. At the extremities of the diameter of a circular garden stands two trees, one 20 feet high and the other 30 feet high. At what point on the circumference must a ladder be placed so that without moving it at the base it will reach to the top of each tree, the diameter of the garden being 40 feet.

Construction.—Let ABC be the circular garden and AC its diameter, and let AF and CD be the two trees at the extremities of the diameter. Connect the tops of the trees by the line FD and from the middle point E of FD let fall the the perpendicular EH. Draw EG perpendicular to FD. Then all points of EG are equally distant from FD. At G draw BG perpendicular to AC. Then all points of BG are equally distant from F and D. Hence, B is the required point. no

II.
1. $AC=2R=40$ft., the diameter of the garden.
2. $CD=a=30$ ft., the height of the tree CD, and
3. $AF=b=20$ ft., the height of the tree AF.
4. $DI=DC-CI\ (=AF)=a-b$ $=40$ ft.-30 ft.$=10$ ft.
5. $EH=\frac{1}{2}(CD+AF)=\frac{1}{2}(a+b)$ $=\frac{1}{2}(40\text{ ft.}+30\text{ ft.})=35$ ft. By similar triangles,
6. $FI:ID::EH:HG$, or $2R:a-b::\frac{1}{2}(a+b):HG$ Whence,

FIG. 74.

7. $HG=\dfrac{a^2-b^2}{4R}=8\frac{3}{4}$ ft.
8. $GB=\sqrt{(BH^2-HG^2)}=\sqrt{\left[R^2-\left(\dfrac{a^2-b^2}{4R}\right)^2\right]}=$ $\dfrac{\sqrt{[16R^4-(a^2-b^2)^2]}}{4R}=\frac{15}{4}\sqrt{23}$ ft.
9. $\therefore AB=\sqrt{(AG^2+GB^2)}=\sqrt{[(AR+HG)^2+(GB^2)]}=$ $\sqrt{\left[\left(R+\dfrac{a^2-b^2}{4R}\right)^2+\dfrac{16R^4-(a^2-b^2)^2}{16R^2}\right]}=5\sqrt{46}$ ft. $=34.91165$ ft., nearly, and

Since $AB^2+BC^2=AC^2$ by Geometry

Then substitution ... $AB=5\sqrt{14^2}$ ... $5\sqrt{12^2}$

10. $BC=\sqrt{(GC^2+GB^2)}=\sqrt{\left[\left(R-\frac{a^2-b^2}{4R}\right)^2+\frac{16R^4-(a^2-b^2)^2}{16R^2}\right]}=\frac{5}{4}\sqrt{82}$ ft.$=11.31942$ ft.

III. $\therefore$ { 34.91165 ft. the distance from the smaller tree, and 11.31942 ft. the distance from the larger tree.

I. Seven men bought a grindstone 5 feet in diameter; what part of the diameter must each grind off so that they may share equally?

Construction.—Let AB be the diameter of the grind stone, O its center, and AO its radius. From A draw any indefinite line AN and on it lay off any convenient unit of length seven times, beginning at A. Let P be the last point of division. Draw OP, and from the other points of division draw lines parallel to OP, in intersecting the radius AO, in the points f, e, d, c, b, and a. Then the radius is divided into seven equal parts. On radius AO, as a diameter describe a semi-circumference AOK, and at a, b, c, d, e, and f, erect perpendiculars intersecting the semi-circumference in M, L, K, I, H, and G. Then with O as a center and radii equal the chords MO, LO, KO, IO, HO, and GO, describe the circles as shown in the figure. Then each man's share will be the area lying between the circumferences of these circles. For, the chord $GO^2=Gf^2+fO^2$ and, by a property of the circle, $Gf^2=Af\times fO$. $\therefore$ $GO^2=Af\times fO+fO^2$, $=(Af+fO)fO=AO\times fO=\frac{1}{7}AO^2$. In like manner $HO^2=AO\times eO=\frac{2}{7}AO^2$, $KO^2=AO\times dO=\frac{3}{7}AO^2$, &c.

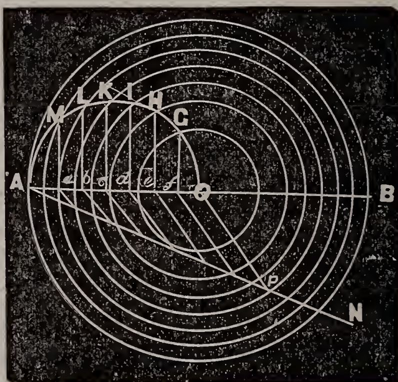


FIG. 75.

1. $AB=D=5$ ft., the diameter of the grind stone.
2. $AO=R=2\frac{1}{2}$ ft., the radius.
3. $\therefore$ $\pi R^2=\pi\times(2\frac{1}{2})^2=6\frac{1}{4}\pi=$the area of the stone.
4. $\frac{1}{7}$ of $\pi R^2=\frac{1}{7}\pi R^2=\frac{1}{7}$ of $6\frac{1}{4}\pi=\frac{25}{28}\pi=$each man's share.
5. $6\frac{1}{4}\pi-\frac{25}{28}\pi=\frac{75}{14}\pi=$the area of the stone after the first has ground off his share.
6. $\therefore$ $\sqrt{(\frac{75}{14}\pi\div\pi)}=\frac{5}{14}\sqrt{42}=2.31455+$ft., the radius MO.
7. $2(AO-MO)=2(2\frac{1}{2}$ ft.$-\frac{5}{14}\sqrt{42}$ ft.$)=2(2\frac{1}{2}$ ft.-2.31455 ft.$)=.3709$ ft., part of the diameter the first grinds off.
8. $6\frac{1}{4}\pi-\frac{2}{7}$ of $6\frac{1}{4}\pi=\frac{125}{28}\pi=$the area after the second grinds off his share.

II.

9. $\therefore \sqrt{(\frac{125}{28}\pi \div \pi)} = \frac{5}{2}\sqrt{\frac{5}{7}} = 2.112875$ ft., the radius LO. Then

10. $2(MO - LO) = 2(5\sqrt{\frac{3}{14}} - \frac{5}{2}\sqrt{\frac{5}{7}}) = 2(2.31455$ ft.$-$ 2.112875 ft.$) = .40335$ ft., the part of the diameter the second grinds off.

11. $6\frac{1}{4}\pi - \frac{3}{7}$ cf $6\frac{1}{4}\pi = \frac{25}{7}\pi =$ the area after the third has ground off his share.

12. $\therefore \sqrt{(\frac{35}{7}\pi \div \pi)} = \frac{5}{2}\sqrt{\frac{4}{7}} = 5\sqrt{\frac{1}{7}} = 1.889822+$ ft., the radius KO. Then,

13. $2(LO - KO) = 2(\frac{5}{2}\sqrt{\frac{5}{7}} - 5\sqrt{\frac{1}{7}}) = 2(2.112875$ ft.$-$ 1.8899822 ft$) = .446106$ ft., the part of the diameter the third grinds off.

14. $6\frac{1}{4}\pi - \frac{4}{7}$ of $6\frac{1}{4}\pi = \frac{75}{28}\pi =$ the area after the fourth has ground off his share.

15. $\therefore \sqrt{(\frac{75}{28}\pi \div \pi)} = \frac{5}{2}\sqrt{\frac{3}{7}} = 1.636634$ ft., the radius IO. Then

16. $2(KO - IO) = 2(5\sqrt{\frac{1}{7}} - \frac{5}{2}\sqrt{\frac{3}{7}}) = 2(1.889822$ ft.$-$ 1.636634 ft.$) = .506376$ ft., the part of the diameter the fourth grinds off.

17. $6\frac{1}{4}\pi - \frac{5}{7}$ of $6\frac{1}{4}\pi = \frac{50}{28}\pi =$ the area after the fifth grinds off his share.

18. $\therefore \sqrt{(\frac{50}{28}\pi \div \pi)} = \frac{5}{2}\sqrt{\frac{2}{7}} = 1.336306$ ft., the radius HO. Then

19. $2(IO - HO) = 2(\frac{5}{2}\sqrt{\frac{3}{7}} - \frac{5}{2}\sqrt{\frac{2}{7}}) = 2(1.636634$ ft.$-$ 1.336306 ft.$) = .600656$ ft., the part of the diameter the fifth grinds off.

20. $6\frac{1}{4}\pi - \frac{6}{7}$ of $6\frac{1}{4}\pi = \frac{25}{28}\pi =$ the area after the sixth grinds off his share.

21. $\therefore \sqrt{(\frac{25}{28}\pi \div \pi)} = \frac{5}{2}\sqrt{\frac{1}{7}} = .949911$ ft., the radius GO. Then

22. $2(HO - GO) = 2(\frac{5}{2}\sqrt{\frac{2}{7}} - \frac{5}{2}\sqrt{\frac{1}{7}}) = 2(1.336306$ ft.$-$ $.944911$ ft.$) = .782790$ ft., the part of the diameter the sixth grinds off.

23. $2 \times .944911$ ft. $= 1.889822$ ft., the diameter of the part belonging to the seventh man.

I. J. A. M., having a wooden ball 2 feet in diameter, bored a hole 1 foot in diamer through the center. What is the volume bored out?

Construction.—Let $ABCDEF$ be a great circle of the ball and let $ACDF$ be a verticle section of the auger hole. Draw the diameter BOE and the radius AO. Then the volume bored out consists of a cylinder, of which $ACDF$ is a vertical section, and two spherical segments, of which ACB and FDE are verticle sections.

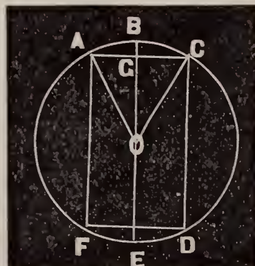


FIG. 76

1. $BE = 2$ feet $= 2R$, the radius of the ball, and
2. $AO = 1$ foot $= 2r$, the radius of the auger hole.

II.
3. $\frac{1}{2}AF=OG=\sqrt{(AO^2-AG^2)}=\sqrt{(R^2-r^2)}=\frac{1}{2}\sqrt{3}$.
4. $\therefore$ $AF=2\sqrt{(R^2-r)^2}=\sqrt{3}$, the length of the cylinder.
5. $\therefore$ $V=\pi r^2\times(\sqrt{3})=\frac{1}{4}\pi\sqrt{3}$, the volnme of the cylinder, and
6. $2V'=2(\frac{1}{2}BG\times\pi AG^2+\frac{1}{6}\pi BG^3)=[R-\sqrt{(R^2-r^2)}]\times\pi r^2+\frac{1}{3}\pi[R-\sqrt{(R^2-r^2)}]^3$
7. $\frac{1}{4}\pi(1-\frac{1}{2}\sqrt{3})+\frac{1}{3}\pi(1-\frac{1}{2}\sqrt{3})^3=\frac{1}{12}\pi(16-9\sqrt{3})$, the volume ot the two spherical segments.
8. $\therefore$ $V+2V'=\frac{1}{4}\pi\sqrt{3}+\frac{1}{12}\pi(16-9\sqrt{3})=\frac{1}{6}\pi(8-3\sqrt{3})$, $=1.46809$ cu. ft.$=2536.85952$ cu. in.

III. $\therefore$ The volume bored out is 2536.85952 cu. in.

I. What is the diameter of the largest circular ring that can be put in a cubical box whose edge is 1 foot?

Construction.—Let $ABCD—E$ be the cubical box. Let I, K, L, M, N, and P, be the middle points of the edge CF, GF, GH, HA, AB, and BC respectively. Connect these points by the lines KI, KL, LH, MN, NP, and PI. Then $IKLMNP$ is a regular hexagon, and the largest ring that can be put in the box will be the inscribed cricle of the hexagon.

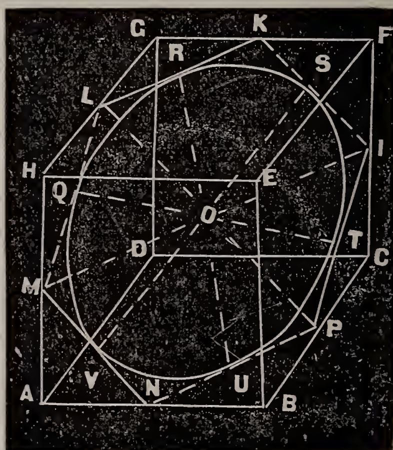


FIG. 77.

II.
1. $AB=12$ in.$=e$, the edge of the cube.
2. $AN=AM=\frac{1}{2}AB=6$ in.$=\frac{1}{2}e$.
3. $\therefore$ $MN=ML=MO=\sqrt{(AN^2+AM^2)}=\sqrt{(2AN^2)}=AN\sqrt{2}=\frac{1}{2}\sqrt{2e}$, the side of the hexagon,
4. $MQ=\frac{1}{2}ML=\frac{1}{2}$ of $\frac{1}{2}\sqrt{2e}=\frac{1}{4}\sqrt{2e}$. Then
5. $OR=\sqrt{(MO^2-MQ^2)}=\sqrt{[(\frac{1}{2}\sqrt{2e})^2-(\frac{1}{4}\sqrt{2e})^2]}=\frac{1}{4}\sqrt{6e}$, the radius of the circle.
6. $\therefore$ $2OR=2\times(\frac{1}{4}\sqrt{6e})=\frac{1}{2}\sqrt{6e}=\frac{1}{2}\sqrt{6}\times12=6\sqrt{6}=$ 14.6969382 in., the diameter.

III. $\therefore$ The diameter of the largest circular ring that can be put in a cubical box whose edge is 1 foot, is 14.6969382 in.

I. A fly takes the shortest route from a lower to the opposite upper corner of a room 18 feet long, 16 feet wide, and 8 feet high. Find the distance the fly travels and locate the point where the fly leaves the floor.

Construction.—Let $FABE—D$ be the room, of which AB is the length, AF the width, and AD the height; and let F be the position of the fly, and C the opposite upper corner to which it is to travel. Conceive the side $ABCD$ to revolve about AB until it comes to a level with the floor and takes the position of $ABC'D'$. Then the shortest path of the fly is the diagonal FC' of the rectangle $FD'C'E$, and P will be the point where the fly leaves the floor.

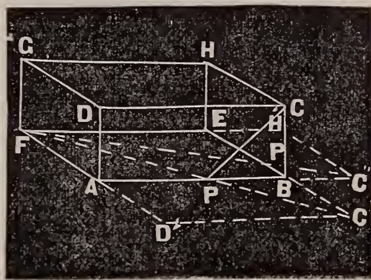


FIG. 78.

II. A. 1. $AB=a=18$ ft., the length of the room,
2. $AF=b=16$ ft., the width, and
3. $AD=h=8$ ft., the height.
4. $FD'=FA+AD'=b+h=16$ ft.$+8$ ft.$=24$ ft. Then
5. $FC'=\sqrt{(FD'^2+D'C'^2)}=\sqrt{[(b+h)^2+a^2]}$, $=\sqrt{[(16+8)^2+18^2]}=30$ feet, the length of the path of the fly.

B. 1. $FD':D'C'::AF:AP$, from the similar triangles $C'D'F$ and PAF, or
2. $b+h\ :a::b:AP$. Whence, $AP=\dfrac{ab}{b+a}=\dfrac{18\times16}{16+8}$ $=12$ feet, the distance from A to where the fly leaves the floor.

III. $\therefore$ 30 feet is the distance the fly travels, and 12 feet is the distance from A to where it leaves the floor.

Remark.—If we conceive the side $BCHE$ to revolve about EH until it is level with the floor, the path of the fly will be FC'' and the length of this is $\sqrt{[(a+h)^2+b^2]}$. But $\sqrt{[(a+h)^2+b^2]}>\sqrt{[(b+h)^2+a^2]}$, because, by expanding the terms under the radicals, it will be seen that the terms are the same, except $2ah$ and $2bh$, and since a is greater than b, FC' is less than FC''. When $a=b$, $FC'=FC''$.

I. How many acres are there in a square tract of land containing as many acres as there are boards in the fence inclosing it, if the boards are 11 feet long and the fence is 4 boards high?

1. $\dfrac{(side)^2}{160}$=number of acres in the tract, the side being expressed in rods.
2. $4\times16\frac{1}{2}\times side$=number of feet in the perimeter of the field.

I. 3. $\therefore 4\times\left[\frac{4\times16\frac{1}{2}\times side}{11}\right]$=number of boards in the fence inclosing the tract.

4. $\therefore \frac{(side)^2}{160}=4\left[\frac{4\times16\frac{1}{2}\times side}{11}\right]=24\times side$. Whence,

5. $(side)^2=160\times24\times side=3840\times side$.

6. $\therefore side=3840$ rods=12 miles.

7. $\therefore (3840)^2\div160=92160$=number of acres in the tract.

III. $\therefore$ There are 92160 A. in the tract.

(*Milne's Pract. Arith., p. 362, prob. 71.*)

SECOND SOLUTION.

II.
1. 16=number of acres comprised between two panels of fence on opposite sides of the field.
2. 1A.=43560 sq. ft.
3. 16 A.=16×43560 sq. ft.=696960 sq. ft.
4. 11ft.=the width of this strip comprised between the two panels.
5. $\therefore$ 12 mi.=63360 ft.=$696960\div11$, the length of the strip, which is the width of the field.
6. 144 sq. mi.=$(12)^2$=the area of the field.
7. 1 sq. mi.=640 A.
8. 144 sq. mi.=144×640 A=92160 A.

III. $\therefore$ There are 92160 A. in the tract.

Explanation—Since for every board in the fence there is an acre of land in the tract for 4 boards, or one panel of fence there would be 4 A. Now a panel on the opposite side of the field would also indicate 4 A. Hence, between two panels on opposite sides of the field there would be comprised a tract 11 ft. wide and containing 8 A. But this would make boards on the *other* two sider of the field have no value. Now the boards on the other two sides having as much value as the boards on the first two sides, it follows that we must take twice the area of the rectangle included between two opposite panels for the area comprised between two opposite panels in the entire tract. Hence, between two opposite panels in the tract there are comprised 16 A. The length of this rectangle is $16\times43560\div11=63360$ ft =12 mi., which the length of the side of the tract.

In any problem of this kind, we may find the length of a side in miles, by multiplying the number of boards in the height of the fence by 33 and divide the product by the length of a board, expressed in feet.

I, How many acres in a circular tract of land, containing as many acres as there are boards in the fence inclosing it, the fence being 5 boards high, the boards 8 feet long, and bending to the arc of a circle?

Construction.—Let C be the center of of the circular tract, $AB=AC=R$, the radius, and the arc $AB=8$ feet. Then the area of the sector is 5 A.=217800 sq. ft.

II.
1. 5 A.$=5\times43560$ sq. ft.$=217800$ sq. ft., the area of the sector ABC.
2. $\frac{1}{2}(AB\times AC)=\frac{1}{2}(8\times AC)=4AC=$area of the sector ABC.
3. $\therefore\ 4AC=217800$ sq. ft. Whence,
4. $AC=217800\div4=54450$ ft.$=3300$ rods, the radius of the circle.
5. $\therefore\ \pi\times(3300)^2\div160=68062.5\pi=$number of acres in tract.

II. $\therefore$ There are 68062.5π A., in the tract.

I. What is the length of a thread wrapped spirally around a cylinder 40 feet high and 2 feet in diameter, the thread passing around 10 times?

II.
1. 2π ft.$=ABCA$ (*Fig. 79*), the circumference of the cylinder
2. 4 ft.$=40$ ft.$\div10=AF$, the distance between the spires.
3. $\sqrt{[(2\pi)^2+4^2]}=2\sqrt{[\pi^2+4]}$ ft.$=AEF$, the length of one spire.
4. $\therefore\ 10\times2\sqrt{[\pi^2+4]}$ft. $=20\sqrt{[\pi^2+4]}$ ft.$=74.4838$ ft., the entire length of the thread.

III. $\therefore$ The entire length of the thread$=74.4838$ ft.

Remark.—Each spire is equivalent to the hypotenuse of a right angled triangle whose base is $ABCA$ and altitude AF. This may be clearly shown by covering a cylinder with paper and tracing the position of the thread upon it. Then cut the paper along the line AFK and spread it upon a plane surface. AEF will then be seen to be the hypotenuse of a right-angled triangle whose base is $ACBA$ and altitude AF.

I. A thread passes spirally around a cylinder 10 feet high and 1 foot in diameter. How far will a mouse travel in unwinding the thread if the distance between the coils is 1 foot?

Construction.—Let $ACB—K$ be a portion of the cylinder and $ADEFGK$ a portioa of the thread. Let A be the position of the mouse when the unwinding begins, P its position at any time afterwards, APN a portion of the path it describes, and PD the portion of the thread unwound. Draw DC parallel to HB and draw OD and OC. Then

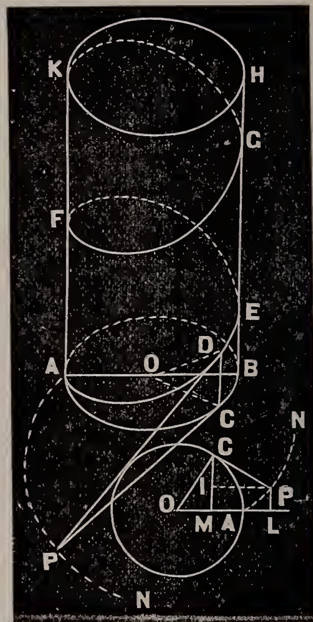


FIG. 79.

II.
1. $AB=2R=1$ foot, the diameter of the cylinder.
2. $a=10$ ft., the altitude. Let
3. $\theta=$the angle AOC,
4. $s=AN$, the length of a portion of the curve,
5. $x=OM$, and
6. $y=PL$. Then
7. $PC=\text{arc } AC=R\theta$,
8. $OM=R\cos\theta$,
9. $ML=IP=CP\cos\angle CPI$
$=R\theta\cos(\frac{1}{2}\pi-\angle PCI)$
$=R\theta\sin\angle PCI=R\theta\sin\theta$.
10. $x=OM+ML=R\cos\theta+R\theta\sin\theta$, and
11. $y=PI=IM=CM-CI$
$=R\sin\theta-CP\cos\theta=$
$R\sin\theta-R\theta\cos\theta$.
12. $dx=R\theta\cos\theta\, d\theta$, by differentiating in 10,
13. $dy=R\theta\sin\theta\, d\theta$, by differentiating in 11. Now
14. $s=\int\sqrt{[dx^2+dy^2]}$.
15. $\therefore$ $s=\int[(R\theta\cos\theta_1 d\theta)^2+(R\theta\sin\theta\, d\theta)^2]^{\frac{1}{2}}=R\int\theta d\theta$
$=\frac{1}{2}R\theta^2$. But
16. $\theta=2\pi$, when one spire is unwound, and
17. $\theta=10\times2\pi=20\pi$, when the unwinding is complete.
18. $\therefore$ $s=\frac{1}{2}R\theta^2=\frac{1}{2}\times\frac{1}{2}(20\pi)^2=100\pi^2=989.96044$ ft., the distance the mouse travels to unwind the thread.

III. $\therefore$ The mouse will travel 989.96044 ft. to unwind the thread.

I. What is the length of a thread winding spirally round a cone, whose radius is R and altitade a, the thread passing round n times and intersecting the slant height at eqnal distances apart?

Let P be any point of the thread, $(x, y, z,)$ the co-ordinates of the point; and, let the angle $PFC(=DOC)=\theta$, $BO=a$, the altitude, $DO=R$, the radius of the base of the cone, and $r=$the radius of the cone at the point P. Then the equations of the thread are: $x=r\cos\theta$ ..... (1), $y=r\sin\theta$ ..... (2), and $z=\frac{a}{2\pi n}\theta$ ....(3). From the similar triangles DEP and DOB, $r=\frac{R}{a}(a-z)=R\left(1-\frac{\theta}{2\pi n}\right)$ ...(4). Now the distance between P and its consecutive position is $\sqrt{(dz^2+dx^2+dy^2)}=$
$\sqrt{\left[1+\left(\frac{dx}{dz}\right)^2+\left(\frac{dy}{dz}\right)^2\right]}dz$. $\therefore$ $s=\int\sqrt{\left[1+\left(\frac{dx}{dz}\right)^2\right.}$

$+\left(\frac{dy}{dz}\right)^2\Big]dz\ldots.(5)$. Substituting the value of r in (1) and (2), and differentiating, we have $dx=-\frac{R}{2\pi n}\Big[\cos\theta+(2\pi n-\theta)\sin\theta\Big]d\theta$ and $dy=-\frac{R}{2\pi n}\Big[\sin\theta-(2\pi n-\theta)\cos\theta\Big]d\theta$.

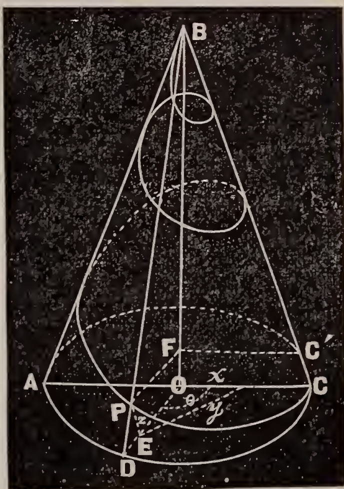


FIG. 80.

From (3), we have $dz=\frac{a}{2\pi n}d\theta$.

Substituting these values of dx, dy, and dz in (5), we have $s=$

$$\int_0^{2\pi n}\frac{a}{2\pi n}\sqrt{\left\{1+\frac{R^2[\cos\theta+(2\pi n-\theta)\sin\theta]^2}{a^2}+\frac{R^2[\sin\theta-(2\pi n-\theta)\cos\theta]^2}{a^2}\right\}}d\theta=$$

$$\int_0^{2\pi n}\frac{a}{2\pi n}\sqrt{\left[1+\frac{R^2}{a^2}+\frac{R^2}{a^2}(2\pi n-\theta)^2\right]}d\theta=\int_0^{2\pi n}\frac{1}{2\pi n}\sqrt{[a^2+R^2+R^2(2\pi n-\theta)^2]}d\theta=\frac{1}{2\pi n}\left\{-\frac{R(2\pi n-\theta)}{2}\sqrt{\left[\frac{a^2+R^2}{R^2}+(2\pi n-\theta)^2\right]}-R\left(\frac{a^2+R^2}{R^2}\right)\log_e\left[(2\pi n-\theta)+\sqrt{\left((2\pi n-\theta)^2+\frac{a^2+R^2}{R^2}\right)}\right]\right\}_0^{2\pi n}=\tfrac{1}{2}\sqrt{(a^2+R^2+4\pi^2n^2R^2)}+\frac{a^2+R^2}{4\pi nR}\log_e\left[\frac{2\pi nR+\sqrt{(a^2+R^2+4\pi^2n^2R^2)}}{\sqrt{(a^2+R^2)}}\right]$$

$$=\tfrac{1}{2}\sqrt{(h^2+4\pi^2n^2R^2)}+\frac{h^2}{4\pi nR}\log_e\left[\frac{2\pi nR+\sqrt{(h^2+4\pi^2n^2R^2)}}{h}\right],$$

where $h=\sqrt{(a^2+R^2)}$, the slant height.

NOTE.—This solution was prepared for the *School Visitor*, by the author.

I. A thread makes n equidistant spiral turns around a cone whose slant height is h, and radius of the base r. The cone stands on a horizontal plane and the string is unwound with the lower end in contact with the plane, the part unwound being always tense. Find the length of the *trace* of the end of the string on the plane.

Let MH be the part unwound at any time, H being the point in contact with the cone, and $BM=u$, the trace on the plane up to this time. Put arc $BE=x$, $AH=y$, E being the point in the circumference of the base in the line AH. Let NI be the posi-

tion of the string at the next instant, D and I being homologous points with E and H. Draw HK parallel to ED. Then $h : DE :: AK : HK$, or $\frac{HK}{ED}=\frac{AK}{h}$ .... (1). Now since the arc BE $=x$, is proportional to the distance the point of contact of the thread with the cone has ascended, $x : h-y :: 2\pi rn : h$, or $\frac{x}{2rn\pi}=\frac{h-y}{h}$. $\therefore\ \frac{dx}{dy}=-\frac{2\pi rn}{y}$ ..... (2). This is negative since y decreases as x increases. It is evident from the figure that $\frac{ED}{IK}=\frac{dx}{dy}=-\frac{2\pi rn.}{h}$ By similar triangles, $IK : HK :: HE : ME$, that is, form (1) and (2), we get $\frac{ME}{h-y}=\frac{HK}{IK}$ $=\frac{ED}{IK}\times\frac{y}{h}=\frac{dx}{dy}\times\frac{y}{h}=-\frac{2\pi rn}{h^2}y$ ..... (3). Therefore, $ME=-\frac{2\pi rn}{h^2}(h-y)y$ ..... (4) Put $ME=t$. Then $\frac{dt}{dy}=-\frac{2\pi rn}{h^2}(h-2y)$ .... (5). By similar figures $r : ME :: ED : MP=\frac{ME\times ED}{r}=-ME\times\frac{2\pi rn}{h}\times IK$.

FIG. 81.

From (3), put $MP=v$, then $\frac{MP}{IK}=\frac{dv}{dy}=\frac{4\pi^2n^2r}{h^3}(h-y)y$ .... (6). Equation (5) gives the entire addition to the line ME which consists of $NP+FD$, since $PF=ME$. Consequently, NP $=\frac{dt}{dy}-\frac{dx}{dy}=-\frac{2\pi rn}{h^2}(h-2y)+\frac{2\pi rn}{h}=\frac{4\pi rn}{h^2}y$ .... (7). Now MN^2 $=MP^2+NP^2$ in the limit. Therefore $\left(\frac{du}{dy}\right)^2=\frac{16\pi^2n^2r^2y^2}{h^4}$ $\left(1+\frac{\pi^2n^2}{h^2}(h-y)^2\right)$ .... (8). $\sqrt{(8)}=(9)$, $\frac{du}{dy}=\frac{4\pi rny}{h^2}\times$ $\sqrt{\left(1+\frac{n^2\pi^2}{h^2}(h-y)^2\right)}$, the intregal of which is u, the length of the trace. Put $h-y=z$, and $\frac{h^2}{n^2\pi^2}=a^2$. Then $u=\frac{4r}{a^2h}\int_o^h(h-z)$ $\sqrt{(a^2+z^2)}dz$ .... (10). Or $u=\frac{4ar}{3h}+\left(\frac{2rh}{3a^2}-\frac{4r}{3h}\right)\sqrt{(a^2+h^2)}$ $+2r\log_e\left[\frac{h+\sqrt{(a^2+h^2)}}{a}\right]$ ..... (11). Write for h, its equal,

$an\pi$, in (11) and we have (12), $u=\frac{4r}{3n\pi}+\frac{2r}{3}\left(n\pi-\frac{2}{n\pi}\right)$
$\sqrt{(1+n^2\pi^2)}+2r\log_e[n\pi+\sqrt{(1+n^2\pi^2)}]$.

This result is independent of h, the cone's slant height, but involves n the number of turns of the thread.

NOTE.—This solution is by Prof. Henry Gunder and is taken from the *School Visitor*, Vol. 9, p. 199. Prof. Gunder stands in the very front rank of Ohio mathematicians. He has contributed some very fine solutions to difficult problems proposed in the *School Visitor* and the *Mathematical Messenger*. He is of a very retiring disposition and does not make any pretentions as a mathematician. But that he possesses superior ability along that line, his solutions to difficult problems will attest. Prof. Gunder was born at Arcanum, O , Sept. 15th, 1837. He passed his boyhood on a farm and it was while following a plow or chopping winter wood, that difficult problems were solved and hitherto unknown fields of thought explored. He became Principal of the Greenville High School in 1867. After seven years' work here, he became Superintendent of the Public Schools of North Manchester, Ind. After five years' work at this place he became Superintendent of schools of New Castle, Ind In 1890, Prof. Gunder was elected professor of Pedagogy in the Findlay, (Ohio) College.

I. A woman printed 10 lbs. of butter in the shape of a right cone whose base is 8 inches and altitude 10 inches. Having company for dinner, she cut off a piece parallel to the altitude and containing $\frac{1}{3}$ of the diameter. What was the weight of the part cut off?

Construction.—Let $ABC—G$ be the cone, AC the diameter and OG the altitude. Let E be the point where the cutting plane intersected the the diameter, F the corresponding point in the slant height, and $DLFKB$ the section formed by the intersection of the cone and the cutting plane. Through F pass a plane parallel to the base ABC and anywhere between this plane and the base, pass a plane $NLMK$. Then,

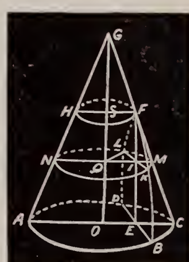


FIG. 82.

1. $AC=2R=8$ in., the diameter of the base,
2. $OG=a=10$ in., the altitude, and
3. $OE=OC-EC=R-\frac{1}{3}AC=R-\frac{2}{3}R=\frac{1}{3}R=1\frac{1}{3}$ in. $=c$, the distance of the cutting plane from the altitude. Let
4 $GQ=x$, the distance of the plane $MLNK$ from the vertex G. By similar triangles,
5. $OC:OG::EC:EF$, or $R:a::R-c:EF$. Whence,
6 $EF=\frac{a(R-c)}{R}=6\frac{2}{3}$ in. By similar triangles,
7. $GO:OC::GQ:QM$, or $a:R::x:QM$. Whence,
8. $QM=LQ=\frac{Rx}{a}$. Now,

9. *area* of LKM=*area* of $LQKM$—*area* of LKQ. But

10. *area* of $LQKM=2\left(\frac{1}{2}LQ^2\cos^{-1}\frac{QI}{LQ}\right)=\left(\frac{Rx}{a}\right)^2\times$ $\cos^{-1}\left(\frac{ac}{Rx}\right)$, and

11. *area* of $LKQ=\frac{1}{2}(LK\times QI)=\frac{1}{2}(2LI\times c)=LI\times c=$ $c\sqrt{(NI\times IM)}=c\sqrt{[(Rx\div a+c)\times(Rx\div a-c)]}=$ $(c\div a)\sqrt{(R^2x^2-c^2a^2)}$.

12. $\therefore$ *Area* of the segment $LKM=\frac{R^2x^2}{a^2}\cos^{-1}\left(\frac{ac}{Rx}\right)-$ $(c\div a)\sqrt{(R^2x^2-c^2a^2)}$.

13. $\left(\frac{R^2x^2}{a^2}\cos^{-1}\left(\frac{ac}{Rx}\right)-\frac{c}{a}\sqrt{(R^2x^2-a^2c^2)}\right)dx=$ an element of volume of the part cut off.

II,

14. $\therefore$ $V=\displaystyle\int_{\frac{ac}{R}}^{a}\left(\frac{R^2x^2}{a^2}\cos^{-1}\left(\frac{ac}{Rx}\right)-\frac{c}{a}\sqrt{(R^2x^2-a^2c^2}\right)dx$

$=\frac{1}{3}a\left\{R^2\cos^{-1}\left(\frac{c}{R}\right)-2c\sqrt{(R^2-c^2)}+\frac{c^3}{R}\times\right.$ $\left.\log_e\left[\frac{R+\sqrt{(R^2-c^2}}{c}\right]\right\}=\frac{10}{3}\left\{4^2\cos^{-1}(\frac{1}{3})-2\times\right.$ $\left.1\frac{1}{3}\sqrt{[4^2-(1\frac{1}{3})^2]}+\frac{1}{27}\times4^2\log_e\left[\frac{4+\sqrt{[4^2-(1\frac{1}{3})^2]}}{1\frac{1}{3}}\right]\right.$,

$=\frac{10}{3}\left\{4^2\cos^{-1}(\frac{1}{3})-\frac{64}{9}\sqrt{2}+\frac{16}{27}\log_e[2+\frac{2}{3}\sqrt{2}]\right\}$,

$=\frac{10}{3}\left\{4^2\times\frac{114257}{291600}\pi-\frac{64}{9}\sqrt{2}+\frac{16}{27}\log_e[2+\frac{2}{3}\sqrt{2}]\right\}$,

$=\frac{10}{3}\left\{19.6938154-10.0562976+.6396202\right\}=$ 34.223792 cu. in., the volume of the part cut off.

15. $\frac{1}{3}a\pi R^2=\frac{1}{3}\times10\times4^2\times\pi=53\frac{1}{3}\pi$ cu. in., the volume of the whole cone.

16. 10 lbs.=the weight of the whole cone. Hence, by proportion,

17. $53\frac{1}{3}\pi$ cu. in.:34.223792 cu. in.::10 lbs.:(?=2.04258 lbs.)

III. $\therefore$ The weigh of the part cut off is 2.04258 lbs.

I. After making a circular excavation 10 feet deep and 6 feet in diameter, it was found necessary to move the center 3 feet to one side; the new excavation being made in the form of a right cone having its base 6 feet in diameter and its apex in the surface of the ground· Reqired the total amount of earth removed.

Construction.—Let $ABC—F$ be the cylindrical excavation first made, AC the diameter, HO the altitude. Let A be the center of the conical excavation, GAH its diameter, and AF, an element of the cylinder, the altitude. Pass a plane at a distance x from O and parallel to the base of the excavation. Let figure II. represent the section thus formed, the letters in this section corresponding to the homologous points in the base represented by the same letters in the base of the excavation. An element of the earth removed in the conical excavation is $(area\ BAKGNB)dx$. The whole volume removed in the conical part of the excavation is

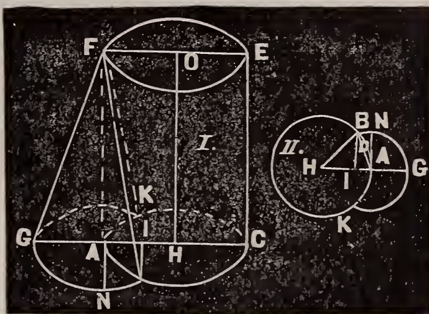


FIG. 83.

$\int_0^a (area\ BAKGNB)dx$. For let

1. $HO=a=10$ ft., the altitude of the excavation,
2. $HA=r=3$ ft., the radius of the cylindrical and the conical parts.
3. $AB=AN=\frac{rx}{a}$. This is found from the proportion of similar triangles.
4. $BI^2=(rx\div a—AI)(rx\div a+AI)$. Also
5. $BI^2=(2r—AI)AI$.
6. $\therefore\ (2r—AI)AI=(rx\div a—AI)(rx\div a+AI)$. Whence,
7. $AI=rx^2\div 2a^2$,
8. $BI=\frac{rx}{2a^2}\sqrt{(4a^2—x^2)}$,
9. $HI=r—\frac{rx^2}{2a^2}=r(1—\frac{x^2}{2a^2})$. Now
10. *area* of $BDAKGNB=2($ *area* of $BDAN+$*area* of $NAG)$. But
11. $\frac{1}{4}\pi(r^2x^2\div a^2)=$ the *area* of the quadrant NAG, and
12. *area* of $BDAN=$*area* of sector $BAN+$ *area* of triangle $HBA—$*area* of sector $BDAH$. Now
13. *area* of sector $BAN=\frac{1}{2}AB\times AB\sin^{-1}(AI\div AB)$
$=(r^2x^2\div 2a^2)\sin^{-1}\left(\frac{x}{2a}\right)$,
14. *area* of triangle $ABH=\frac{1}{2}(AH\times BI)=\frac{1}{2}r\times(rx\div 2a^2)\times\sqrt{(4a^2—x^2)}=(r^2x\div 4a^2)\sqrt{(4a^2—x^2)}$, and
15. *area* of sector $BDAH=\frac{1}{2}[AH\times AH\cos^{-1}(HI\div BH)]$
$=\frac{1}{2}r^2\times\cos^{-1}[1—(x\div 2a)]$.

II. 16. $\therefore$ *Area* of $BDAKGNB=2\left\{\frac{1}{4}\frac{r^2x^2}{a^2}\pi+\frac{r^2x^2}{2a^2}\sin^{-1}\left(\frac{x}{2a}\right)+\frac{r^2x}{4a^2}\sqrt{(4a^2-x^2)}-\frac{1}{2}r^2\cos^{-1}\left(1-\frac{x^2}{2a^2}\right)\right\}$

$=\frac{r^2x^2}{2a^2}\pi+\frac{r^2x^2}{a^2}\sin^{-1}\left(\frac{x}{2a}\right)+\frac{r^2x}{2a^2}\sqrt{(4a^2-x^2)}$

$-r^2\cos_{-1}\left(1-\frac{x^2}{2a^2}\right)$.

17. $\therefore$ $V=\int_o^a\left\{\frac{r^2x^2}{2a^2}\pi+\frac{r^2x^2}{a^2}\sin^{-1}\left(\frac{x}{2a}\right)+\frac{r^2x}{2a^2}\sqrt{(4a^2-x^2)}-r^2\cos^{-1}\left(1-\frac{x^2}{2a^2}\right)\right\}dx=\frac{1}{6}\pi ar^2+\frac{r^2}{a^2}\int_o^a x^2\sin^{-1}\left(\frac{x}{2a}\right)dx+\frac{r^2}{2a^2}\int_o^a x^2\sqrt{(4a^2-x^2)}dx-r^2\int_o^a\cos^{-1}\left(1-\frac{x^2}{2a^2}\right)dx=\frac{1}{6}\pi ar^2+\frac{r^2}{a^2}\left[\frac{1}{3}x^3\sin^{-1}\frac{x}{2a}+\frac{1}{9}(x^2+8a^2)(4a^2-x^2)^{\frac{1}{2}}\right]_o^a-\frac{r^2}{2a^2}\left[\frac{1}{3}(4a^2-x^2)^{\frac{3}{2}}\right]_o^a-r^2\left[x\cos^{-1}\left(1-\frac{x^2}{2r^2}\right)+2(4a^2-x^2)^{\frac{1}{2}}\right]_o^a=\left(\frac{64-27\sqrt{3}-2\pi}{18}\right)ar^2$,

=the volume of the cylindrical part of the excavation.

18. πar^2=the volume of the conical part.

19. $\therefore$ $\pi ar^2+\left(\frac{64-27\sqrt{3}-2\pi}{18}\right)ar^2=\left(\frac{64-27\sqrt{3}+16\pi}{18}\right)\times ar^2=337.500554$ cu. ft., the volume of the entire excavation.

III. $\therefore$ The volume of the excavation $=\left(\frac{64-27\sqrt{3}+16\pi}{18}\right)ar^2$,

or 337.50055+cu. ft., correct to the last decimal place.

NOTE.—This problem was proposed in the *School Visitor* by Wayland Dowling, Rome Center, Mich. A solution to the problem, by Henry Gunder, was published in Vol. 9, No. 6, p. 121. The solution there given is by polar coordinates. The editor gives the answers obtained by the contributors; viz., Mr. Dowling, H. A. Wood, R. A. Leisy, and William Hoover. Their answers differ from Mr. Gunder's and from each other. Mr. Gunder's answer is 337.5+cu. ft., the same as above. There is a similar problem in *Todhunter's Integral Calculus, p. 190, prob. 29.*

34

I. A tree 74 feet high, standing perpendicularly, on a hill-side, was broken by the wind but not severed, and the top fell directly down the hill, striking the ground 18 feet from the root of the tree, the horizontal distance from the root to the broken part being 18 feet, find the height of the stub.

Construction.—Let AD be the hill-side, AB the stump, BD

the broken part, and AC the horizontal line from the root of the tree to the broken part. Produce AB to E and draw DE parallel to AC.

II.

1. Let $AB=x$, the height of the stump. Then
2. $BD=74$ ft.$-x=s-x$, the broken part, since $AB+BD=74$ feet.
3. Let $AD=a=34$ ft., the distance from the foot of the tree to where the top struck the ground,
4. $AC=b=18$ ft., the horizontal distance from the foot of the tree to the broken part.
5. $x=AB$, the height of the stump. Then
6. $BC=\sqrt{(AB^2+AC^2)}=\sqrt{(x^2+b^2)}$ . . (1). In the simlar triangles BAC and BED,
7. $\sqrt{(x^2+b^2)} : x :: s-x : BE$. Whence,
8. $BE=\dfrac{x(s-x)}{\sqrt{(x^2+b^2)}}$ . . . .(2). Also,
9. $\sqrt{(x^2+b^2)} : b :: s-x : DE$. Whence
10. $DE=\dfrac{b(s-x)}{\sqrt{(x^2+b^2)}}$ . . . (3). Now
11. $AE=BE-BA=\dfrac{x(s-x)}{\sqrt{(x^2+b^2)}}-x$ . . . . . (4).
12. $AE^2+ED^2=AD^2$, or
13. $\left\{\dfrac{x(s-x)}{\sqrt{(x^2+b^2)}}-x\right\}^2+\left\{\dfrac{b(s-x)}{\sqrt{(x^2+b^2}}\right\}^2=a^2$ . . . (5). Developing (5), we have
14. $4(s^3-a^2+b^2)x^4-4s(s^2-a^2+2b^2)x^3+(s^4+a^4-2a^2s^2+8b^2s^2-4a^2b^2)x^2-4b^2s(s^2-a^2)x=-b^2(s^2-a^2)$ . . . . (6).
15. $1161x^4-91908x^3+1959876x^2-25894080x+377913600=0$ . . . . . (7), by substituting the values of a, b, and s in (6).
16. $\therefore x=24$ feet, the height of the stump, by solving (7) by Horner's method.

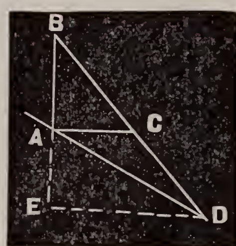


FIG. 84.

III. $\therefore$ The height of the stump is 24 feet.

NOTE.—This problem was taken from the *Mathematical Magazine*, Vol. I., No. 7, prob. 84. In Vol I., p. 184, of the *Mathematical Magazine* is a solution to it, given by C. H. Scharar and Prof. J. F. W. Sheffer The solution there given is different from the one above.

I. What is the longest strip of carpet one yard wide that can be laid diagonally in a room 30 feet long and 20 feet wide?

Construction.—Let $ABCD$ represent the room and $EFGH$ the strip of carpet one yard wide placed diagonally in the room.

II.

1. Let $AB=a=30$ ft., the length of the room,
2. $BC=b=20$ ft., the width, and
3. $HG=c=3$ ft., the width of the carpet. Let
4. $BF=HD=x$. Then
5. $FC=AH=20-x=b-x$.
6. $BE=\sqrt{(EF^2-BF^2)}=\sqrt{(9-x^2)}=\sqrt{(c^2-x^2)}$ ..(1),
7. $AE=GC=AB-EB=a-\sqrt{(c^2-x^2)}$ .... (2). By similar triangles,
8. $EF:BF::GF:GC$, or
9. $c:x::GF:a-\sqrt{(c^2-x^2.)}$ Whence,
10. $GF=\dfrac{c[a-\sqrt{(c^2-x^2)}]}{x}$...3

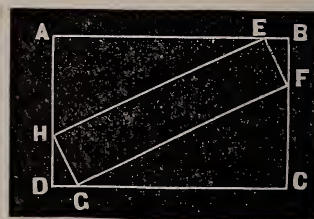


FIG. 85.

Again, we have

11. $EF:BE::GF:FC$, or
12. $c:\sqrt{(c^2-x^2)}::GF:b-x$.
13. $\therefore GF=\dfrac{c(b-x)}{\sqrt{(c^2-x^2)}}$ ...(4). By equating GF in (3) and (4),
14. $\dfrac{c(b-x)}{\sqrt{(c^2-x^2)}}=\dfrac{c[a-\sqrt{(c^2-x^2)}]}{x}$... (5).
15. $bx-x^2=a\sqrt{(c^2-x^2)}-c^2+x^2$ ...(6), by dividing (5) by c and clearing of fractions.
16. $c^2-bx-2x^2=a\sqrt{(c^2-x^2)}$...(7), by transposing in (6).
17. $4x^4-4bx^3+(a^2+b^2-4c^2)x^2+2bc^2x=c^2(a^2-c^2)$...-...(8), by squaring (7) and transposing and combining.
18. $4x^4-80x^3+1264x^2+360x=8019$ .... (9), by restoring numbers in (8).
19. $\therefore x=2.5571+$ft., by solving (9) by Horner's method.
20. $\therefore\sqrt{(c^2-x^2)}=\sqrt{(9-x^2)}=1.5689$ ft. Then,
21. $GC=30-\sqrt{(9-x^2)}=28.4311$ ft., and
22. $FC=20-x=17.4429$ ft.
23. $\therefore GF=\sqrt{(FC^2+GC^2)}=\sqrt{[(28.4311)^2+(17.4229)^2]}$ $=33.3554$ ft., the length of the carpet.

III. $\therefore$ The length of the strip of carpet is 33.3554 ft.

I. What length of rope, fastened to a point in the circumference of a circular field whose area is one acre, will allow a horse to graze upon just one acre outside the field?

Construction.—Let $ABPC$ be the circular field and P the point in the circumference to which the horse is fastened. Let BP represent the length of the required rope. Draw the radius BO of the field and the line BC. Then

1. 1 A.=160 sq. rd.=the area of the field $ABPC$, and
2. $BO=OP=R=\sqrt{(160\div\pi)}=4\sqrt{\left(\dfrac{10}{\pi}\right)}$, the radius of

the circular field. Let

3. θ=the angle BPO=the angle OBP. Hence,
4. $\pi-2\theta$=the angle BOP. Now
5. $BP=AP\cos\angle APB=2R\cos\theta$, the length of the required rope. The
6. *area* $BPCD$ over which the horse grazes=*area* $BECDB$—*area* $BECPB$. But
7. *area* of circle $BECD=\pi BP^2=\pi 4R^2\cos^2\theta=4\pi R^2\cos^2\theta$, and the
8. *area* $BECP=2\times($*area* of sector EPB+*area* of segment $BPH)$. Now

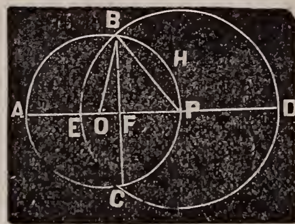


FIG. 86.

9. *area* of sector $EPB=\frac{1}{2}BP\times$ *arc* $BE=\frac{1}{2}\times2R\cos\theta\times2R\cos\theta\times\theta=2R^2\theta\cos^2\theta$, and
10. *area* of segment BPH=*area* of sector BOP—*area* of triangle $OBP=\frac{1}{2}BO\times arcBHP-\frac{1}{2}OP\times BF=\frac{1}{2}[R\times R(\pi-2\theta)]-\frac{1}{2}R\times R\sin(\pi-2\theta)$, $=\frac{1}{2}R^2(\pi-2\theta)-\frac{1}{2}R^2\sin2\theta$.

II.

11. $\therefore$ *Area* $BECD=2[2R^2\theta\cos^2\theta+\frac{1}{2}R^2(\pi-2\theta)-\frac{1}{2}R^2\sin2\theta]=R^2[4\theta\cos^2\theta+\pi-2\theta-\sin2\theta]=R^2[4\theta\left(\frac{1+\cos2\theta}{2}\right)+\pi-2\theta-\sin2\theta]=R^2[\pi+2\theta\cos2\theta-\sin^22\theta]$.
12. $\therefore$ *Area* $BPCDB=4\pi R^2\cos^2\theta-R^2[\pi+2\theta\cos2\theta-\sin2\theta]$. But
13. πR^2=1A.=160 sq. rd.=the area of $BPCDB$, by the conditions of the problem.
14. $\therefore$ $4\pi R^2\cos^2\theta-R^2[\pi+2\theta\cos2\theta-\sin2\theta]=\pi R^2$. Whence,
15. $4\pi\left(\frac{1+\cos2\theta}{2}\right)-[\pi+2\theta\cos2\theta-\sin2\theta]=\pi$, or
16. $2\pi+2\pi\cos2\theta-\pi-2\theta\cos2\theta+\sin2\theta=\pi$.
17. $\therefore$ $2\theta\cos2\theta-\sin2\theta=2\pi\cos2\theta$, or
18. $2\theta-\tan2\theta=2\pi$, by dividing by $\cos2\theta$. Whence,
19. $\theta=51°\ 16'\ 24''$, by solving the last equation by the method of Double Position.
20. $\therefore$ $BP=2R\cos\theta=8\sqrt{\left(\frac{10}{\pi}\right)}\cos''=8.92926+$rods.

III. $\therefore$ The length of the rope is 8.92926+rods.

I. If a 2-inch auger hole be bored diagonally through a 4-inch cube, what will be the volume bored out, the axis of the auger hole coinciding with the diagonal of the cube?

Formula.— $V=r^2\sqrt{3}(\pi e-2r\sqrt{2})$, where e is the edge.

Coustrüction.—Let $AFGD$ be the cube and DF the diagonal, which is also the axis of the auger hole. The volume bored out will consist of two equal tetrahedrons $acd—D$ and $efg—F$ *plus* the cylinder $acd—f$, *minus* 6 cylindrical ungulas each equal to $ace—b$. Pass a plane any where between e and b, perpendicular to the axis of the cylinder, and let x be the distance the plane is from D. Now let

II.

1. $AB=e=4$ inches, the edge of the cube;
2. $DF=\sqrt{3}s=4\sqrt{3}$, the diagonal of the cube; and
3. $r=1$ inch, the radius of the auger, or the radius of the circle acd.

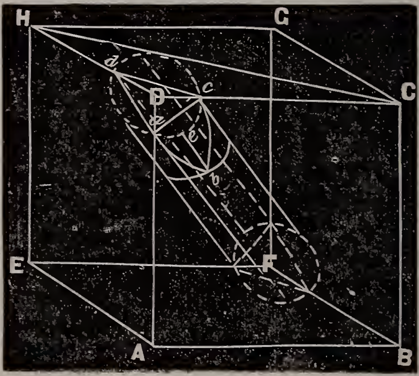


FIG. 87

4. $ac=ad=dc=r\sqrt{3}$ $=\sqrt{3}$.
5. $Dc=\frac{1}{2}r\sqrt{6}=\frac{1}{2}\sqrt{6}$, by the similar triangles dDc and HDc.
6. $\sqrt{(Dc^2-r^2)}=\sqrt{[(\frac{1}{2}r\sqrt{6})^2-r^2]}=\frac{1}{2}r\sqrt{2}=\frac{1}{2}\sqrt{2}$, the altitude of the tetrahedron $acd—D$.
7. $\therefore\ 2v=\frac{2}{3}(\textit{area}\text{ of base}\times\text{altitude})=2(\frac{1}{4}\sqrt{3}\times ac^2\times\frac{1}{3}\times\frac{1}{2}r\sqrt{2})=\frac{1}{4}\sqrt{6}r^3=\frac{1}{4}\sqrt{6}$, the volume of the two tetrahedrons,
8. $v'=\pi r^2\times(DF-2\text{ times the }\textit{altitude}\text{ of }acd—D)=\pi r^2(e\sqrt{3}-\frac{1}{2}r\sqrt{2})=\pi(4\sqrt{3}-\frac{1}{2}\sqrt{2})$, the volume of the cylinder $acd—f$.
9. $be=\frac{1}{2}r\sqrt{2}$, by similar triangles, not shown in the figure.
10. $\frac{1}{2}r\sqrt{2}+\frac{1}{2}r\sqrt{2}=r\sqrt{2}=$distance from D to where the auger begins to cut an entire circle.
11. $r-\frac{1}{2}x\sqrt{2}=$*versin* of an arc of the ungulas at a distance x from D.
12. $2r\cos^{-1}\left(\dfrac{\frac{1}{2}x\sqrt{2}}{r}\right)=$an *arc* of the ungulas at a distance x from D.
13. $r^2\cos^{-1}\left(\dfrac{\frac{1}{2}x\sqrt{2}}{r}\right)-\frac{1}{2}x\sqrt{2}(r^2-\frac{1}{2}x^2)^{\frac{1}{2}}=$the area of a segment at a distance x from D.
14. $\therefore\ 6v''=6\displaystyle\int_{\frac{1}{2}r\sqrt{2}}^{r\sqrt{2}}\left[2r^2\cos^{-1}\left(\frac{\frac{1}{2}x\sqrt{2}}{r}\right)\frac{1}{2}x\sqrt{2}(r^2-\frac{1}{2}x^2)^{\frac{1}{2}}\right]dx$

$$=6\left[r^2x\cos^{-1}\left(\frac{\frac{1}{2}\sqrt{2}x}{r}\right)-\sqrt{2}r^3\sqrt{\left(1-\frac{x^2}{2r^2}\right)}+\right.$$

$\frac{1}{3}\sqrt{2}(r^2-\frac{1}{2}x^2)^{\frac{3}{2}}\Big]_{\frac{1}{2}\sqrt{2}r}^{\sqrt{2}r}=6r^3(\frac{3}{8}\sqrt{6}-\frac{1}{6}\pi\sqrt{2})=$
$r^3(\frac{9}{4}\sqrt{6}-\pi\sqrt{2})$.

15. $\therefore$ V, the volume bored out, $=2v+v'-6v''=\frac{1}{4}\sqrt{6}r^3+\pi r^2(e\sqrt{3}-\frac{1}{2}r\sqrt{2})-r^3(\frac{9}{4}\sqrt{6}-\pi\sqrt{2})=r^2\sqrt{3}(\pi c-2r\sqrt{2})$ $=16.866105$ cu. in.

III. $\therefore$ The volume bored out is 16.866105 cu. in.

I. A horse is tethered to the outside of a circular corral. The length of the tether is equal to the circumference of the corral. Required the radius of the corral supposing the horse to have the liberty of grazing an acre of grass.

Coustruction.—Let $AEFBK$ be the circular corral, AB the diameter, and A the point where the horse is tethered. Suppose the horse winds the tether around the entire corral; he will then be at A. If he unwinds the tether, keeping it stretched, he will describe an involute, $APGH'$, to the corral. From H' to H, he will describe a semi-circle, radius $AH'=AH=$ to the circumference of the corral. From H through G to A, he will again describe an involute.

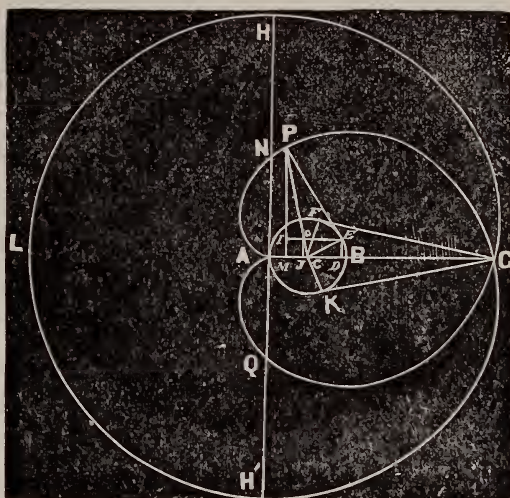


FIG. 88.

Then the area over which he grazes is the semi-circle $HLH'+$ the two equal involute areas $AFGHA$ and $AKGH'A+$the area $BFGKB$.

Let C be the center of the corral and also the origin of co-ordinates, AG the x–axis and P any point in the curve $APGH'$.

1. Let θ=the angle ACE that the radius CE perpendicular to PE, the radius of curvature of the curve $APGH'$, makes with the x-axis,
2. θ_0=the angle $AFEBK$ that the radius CK makes with the x–axis when the radius of curvature PE has moved to the position KG;
3. $R=AC$, the radius of the corral;
4. $\rho=PE=arc\ AFE=R\theta$, the radius of curvature of

the involute;
5. $x=CM$ and
6. $y=PM$, the co-ordinates of the point P; and
7. $x_0=CG$ and
8. $y_0=0$, the co-ordinates of the point G. Then we have
9. $x=CM=IE-CD=PE(=arc\ AFE)\cos\angle IEP$, $=\angle PEC-\angle OEC(=\angle ECD),-CE\cos\angle EGD$ $=R\theta\cos(\angle IEP-\angle ECD)-R\cos(\pi-\theta)=R\theta\cos[\frac{1}{2}\pi-(\pi-\theta)]-R\cos(\pi-\theta)=R\theta\cos-(\frac{1}{2}\pi-\theta)$ $-R\cos(\pi-\theta)=R\theta\cos\theta+R\sin\theta\ldots.(1)$.
10. $y=PM=PI+IM(=DE)=PE\sin\angle PEI+EC\times\sin\angle ECD=R\theta\sin(\theta-\frac{1}{2}\pi)+R\sin(\pi-\theta)=R\sin\theta$ $-R\theta\sin\theta\ldots.(2)$. When $\theta=\theta_0$=angle $AFEBK$

II.
11. $x_0=CG=R\cos\theta_0+R\theta_0\sin\theta_0\ldots.(3)$, and
12. $y_0=0=R\sin\theta_0-R\theta_0\cos\theta_0\ldots.(4)$. Hence, from (4),
13. $\theta_0=R\sin\theta_0\div R\cos\theta_0=\tan\theta_0\ldots.(5)$. Then, from (3),
14. $x_0=R\cos\theta_0+R\tan\theta_0\sin\theta_0=R\left(\cos\theta_0+\dfrac{\sin\theta_0}{\cos\theta_0}\sin\theta_0\right)$ $=\dfrac{R}{\cos\theta_0}=R\sec\theta_0=R\sqrt{(1+\tan^2\theta_0)}=R\sqrt{(1+\theta_0^2)}\ldots.$ (5). Now
15. $BFGKB=2[\frac{1}{2}KG\times KC-\text{sector }BCK]=R^2\theta_0-R^2(\theta_0-\pi)\ldots.(7)$,
16. $AFGHA+AKGH'=2\int dA=2\int\frac{1}{2}\rho^2d\theta=\int_{\theta_0}^{2\pi\theta}R^2\theta^2d\theta$ $=\frac{1}{3}R^2(8\pi-\theta_0^3)\ldots.(8)$, and
17. $HH'L=\frac{1}{2}\pi(AH)^2=\frac{1}{2}\pi(2\pi R)^2=2\pi^3R^2\ldots.(9)$. Adding (7), (8), and (9),
18. $R^2\theta_0-R^2(\theta-\pi)+\frac{1}{3}R^2(8\pi-\theta_0^3)+2\pi^3R^2=$ $R^2(\pi+\frac{14}{3}\pi^3-\frac{1}{3}\theta_0^3)$=area over which the horse grazes.
19. 1 A.=160 sq. rd.=43560 sq. ft.=the area over which the horse grazes.
20 $\therefore R^2(\pi+\frac{14}{3}\pi^3-\frac{1}{3}\theta_0^3)=43560$ sq. ft. Whence,
21 $R=\sqrt{\left(\dfrac{43560}{\pi+\frac{14}{3}\pi^3-\frac{1}{3}\theta_0^3}\right)}\ldots.(10)$. But
22. $\theta_0=4.494039=264^\circ\ 37'\ 18''.35$ by solving (5) by the method of Double Position.
23. $\therefore R=19.24738$ ft., by substituting the value of θ_0 in(10).

III. $\therefore$ The radius of the corral is 19.24738 ft.

A 20-foot pole stands plumb against a perpendicular wall. A cat starts to climb the pole, but for each foot it ascends the pole slides one foot from the wall; so that when the top of the pole is reached, the pole is on the ground at right angles to the wall. Required the equation to the curve the cat described and the distance through which it traveled.

Construction.=Let AC be the wall, P the position of the cat at any time, and BC the position of the ladder at the same time. Draw AP and to the middle point D of AP draw BD. Then $AB=PB$.

II. A.

1. Let $BC=20$ ft.$=a$, the length of the ladder,
2. $AP=r$, the radius vector of the curve the cat describes, and
3. $\theta=$the angle PAB.
4. $\pi-2\theta=$the angle ABP, because the angle $PAB=$the angle BPA.
5. $AB=BC\cos ABC=a\cos(\pi-2\theta)=-a\cos 2\theta$,
6. $\frac{1}{2}AP=\frac{1}{2}r=AD=AB\cos\angle BAD=-a\cos 2\theta\cos\theta$.
7. $\therefore r=-2a\cos 2\theta\cos\theta$, or
8. $r+2a\cos 2\theta\cos\theta=0$, the equation of the curve described by the cat.

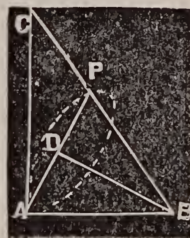


FIG. 89.

B.

1. Let $s=$the distance through which the cat traveled.
2. $s=\int\sqrt{(dr^2+r^2d\theta^2)}=2a\int_0^{\frac{1}{2}\pi}\sqrt{(1-12\cos^2\theta+44\cos^4\theta-32\cos^6\theta)}d\theta$,
3. $=-a\int_0^{\frac{1}{2}\pi}\sqrt{(2-4\cos\phi-\cos^2\phi+4\cos^3\phi)}d\phi$. where $\phi=\pi-2\theta$,
4. $=-\frac{1}{2}a\int_0^{\frac{1}{2}\pi}\sqrt{(6-4\cos\phi-2\cos^2\phi+4\cos^3\phi)}d\phi$ $=1.1193\,a$,
5. $=22.386$ ft., the distance through which the cat travels.

III. $\therefore$ $r+2a\cos 2\theta\cos\theta=0$, is the equation or the curve, and 22.386 ft.=the distance through which cat traveled.

NOTE.—The integration in this problem is performed by Cote s' Method of Approximation.

I. Suppose W. A. Snyder builds a coke oven on a circular bottom 10 feet in diameter. While building it, he keeps one end of a pole 10 feet long, always against the place he is working and the other end in that point of the circumference of the bottom opposite him. Required the capacity of the oven.

Construction.—Let AB be the diameter of the base and CG the altitude. At a distance x from the base pass a plane intersecting the oven in F and E. Draw AE and AC.

II.
1. $AB=2R=10$ feet, the diameter of the base.
2. $AC=AE=2R=10$ feet, by conditions of the problem.
3. $CG=\sqrt{(AC^2-AG^2)}=(4R^2-R^2)=R\sqrt{3}$, the altitude.
4. $EH^2=x^2=(3AG+GH)(GB-GH)=3AG^2-2AG\times GH=3R^2-2R\times GH(=EI)$, because EH is the ordinate of a semi-circle whose diameter is $2AB$. From this, we find
5. $EI=\sqrt{(4R^2-x^2)}-R$. Then
6. $\pi EI^2=\pi[\sqrt{(4R^2-x^2)}-R]^2$, the area of the circle whose center is I.
7. $\therefore V=\int_0^{R\sqrt{3}}\pi[\sqrt{(4R^2-x^2)}-R]^2dx=\int_0^{R\sqrt{3}}[5R^2-x^2-2R\sqrt{(4R^2-x^2)}]dx$,
8. $=\left[5R^2x-\frac{1}{3}x^2-4R^3\sin^{-1}\frac{x}{2R}-Rx\sqrt{(4R^2-x^2)}\right]_0^{R\sqrt{3}}$,
9. $=\frac{1}{3}\pi R^2(9\sqrt{3}-4\pi)=\frac{1}{3}\pi 5^3(9\sqrt{3}-4\pi)$,
10. $=\frac{1}{3}\pi 125(9\sqrt{3}-4\pi)=395.590202+$ cu. ft.

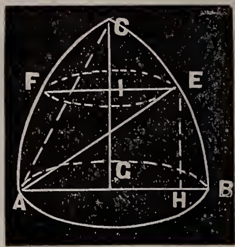


FIG 90.

III. $\therefore$ The capacity is 395.490202 cu. ft.

I. At each corner of a square field whose sides are 10 rods, a horse is tied with a rope 10 rods long; what is the area of the part common to the four horses?

Construction.—Let $ABCD$ be the field and $EFGH$ the area common to the four horses. Join EF, EG, GH, and EH. Draw DK perpeudicular to EF and draw DE and DF. Since $AF=DE=DK=GF=CE$, the triangles ADF and EDC are equilateral and, consequently, the angle $ADE=\angle ADC-\angle EDC=90^\circ-60^\circ=30^\circ$. Also the angle $FCD=30^\circ$. Hence, $EDF=30^\circ$. Now let

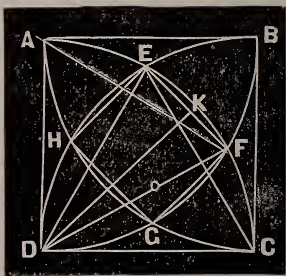


FIG. 91.

II.
1. $AD=ED=a=10$ rods. Then
2. Area of sector $EDF=\frac{1}{2}ED\times$ *arc* $EKF=\frac{1}{2}[a\times(2\pi a)\times\frac{30}{360}]=\frac{1}{12}\pi a^2$.
3. *Area* of triangle $EDF=\frac{1}{2}EF\times DK$. But,
4. $\frac{1}{2}EF=\frac{1}{2}\sqrt{a[2a-\sqrt{(4a^2-AF^2)}]}$, by formula of Prob. XXII., $=\frac{1}{2}a\sqrt{(2-\sqrt{3})}$, and
5. $DK=\sqrt{[DE^2-(\frac{1}{2}EF)^2]}=\frac{1}{4}a\sqrt{(2+\sqrt{3})}$. Hence,
6. *area* of triangle $EDF=\frac{1}{2}a\sqrt{(2+\sqrt{3})}\times\frac{1}{2}a\sqrt{(2-\sqrt{3})}=\frac{1}{4}a^2$.

7. $\therefore$ *Area* of segment $EF=\frac{1}{12}\pi a^2-\frac{1}{4}a^2=\frac{1}{12}a^2(\pi-3)$. The
8. *area* of square $EFGH=EF^2=a^2(2-\sqrt{3})$. Hence,
9. *area* of the figure $EFGH=a^2(2-\sqrt{3})+4\times\frac{1}{12}a^2(\pi-3)$ $=a^2(\frac{1}{3}\pi+1-\sqrt{3})=31.5147$ sq. rd.=the area common to the four horses.

III. $\therefore$ The area of the part common to the four horses is 31.5147 sq. rd.

NOTE.—This problem is similar to problem 348, *School Visitor*, to which a fine trigonometrical solution is given by Prof. E. B. Seitz.

I. What is the length of the longest straight, inflexible stick of wood that can be thrust up a chimney, the arch being 4 feet high and 2 feet from the arch to the back of the chimney—the back of the chimney being perpendicular?

Construction.—Let $PDEC$ be a verticle section of the chimney, PB the height of the arch, PE the distance from the arch to the back of chimney, and APD the longest stick of wood that can be thrust up the chimney.

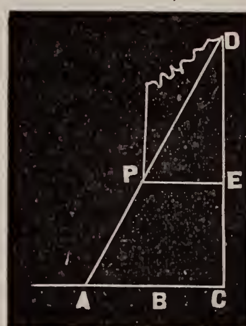


FIG. 92.

II.
1. Let $PB=a=4$ feet, the height of the arch,
2. $PE=b=2$ feet, the width of the chimney,
3. x=the length of the longest stick of wood, and
4. θ=the angle DAC. Then
5. $AP=PB\operatorname{cosec}\theta=a\operatorname{cosec}\theta$,
6. $PD=PE\sec\theta=b\sec\theta$.
7. $\therefore x=AP+PD=a\operatorname{cosec}\theta+b\sec\theta \ldots\ldots (1)$. Differentiating (1),
8. $0=-a\cos\theta\div\sin^2\theta+b\sin\theta\div\cos^2\theta \ldots\ldots (2)$, or
9. $a\cos^3\theta=b\sin^3\theta \ldots\ldots (3)$, by clearing of fractions and transposing in (2).
10. $\therefore \frac{\sin^3\theta}{\cos^3\theta}=\tan^3\theta=\frac{a}{b}$. Whence,
11. $\tan\theta=\sqrt[3]{\frac{a}{b}}$. From (3), we may also have
12. $\cot\theta=\sqrt[3]{\frac{b}{a}}$. Now, from trigonometry,
13. $\sqrt{(1+\tan^2\theta)}=\sec\theta$, and
14. $\sqrt{(1+\cot^2\theta)}=\operatorname{cosec}\theta$. Hence, by substituting in (1),
15. $x=a\sqrt{(1+\cot^2\theta)}+b\sqrt{(1+\tan^2\theta)}=$

$$a\sqrt{\left[1+\left(\frac{b}{a}\right)^{\frac{2}{3}}\right]}+b\sqrt{\left[1+\left(\frac{a}{b}\right)^{\frac{2}{3}}\right]}=a^{\frac{2}{3}}\sqrt{(a^{\frac{2}{3}}+b^{\frac{2}{3}})}$$

$$+b^{\frac{2}{3}}\sqrt{(a^{\frac{2}{3}}+b^{\frac{2}{3}})}=(a^{\frac{2}{3}}+b^{\frac{2}{3}})(\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}})=(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}},$$

$$=\sqrt{[(a^{\frac{2}{3}}+b^{\frac{2}{3}})^3]}=\sqrt{[(4^{\frac{2}{3}}+2^{\frac{2}{3}})^3]}=8.323876+\text{ft.}$$

III. $\therefore$ The length of the longest stick is 8.323876+ft.

I. A small garden, situated in a level plane is surrounded by a wall having twelve equal sides, in the center of which are twelve gates. Through these and from the center of the garden 12 paths lead off through the plane in a straight direction. From a point in the path leading north and at a distance of 4 furlongs $47\frac{19}{243}$ yards from the center of the garden, A. and B. start to travel in opposite directions and at the same rate. A. continues in the direction he first takes; B., after arriving at the first road (lying east of him) by a straight line and at right angles with it, turns so as to arrive at the next path by a straight line and at right angles with it and so on in like manner until he arrives at the same road from which he started, having made a complete revolution around the center of the garden. At the moment that B. has performed the revolution, how far will A. and B. be apart?

Let O be the center of the garden, A the point in the path leading north from which A. and B. start, $C, D, E, F, G, H, I, K, L, M, N, P$, the points at which B. strikes the paths. The triangles OCA, ODC, OED, OFE, &c., are right triangles, OCA, ODC, OED, OEF, &c., being the right angles. Let S in the prolongation of AC denote the position of A., when B., arrives at P. It is required to find the distance AS. Let $OA=a=4$ furlongs, $47\frac{19}{243}$ yd., $AS=AC+CD+DE+\ldots+NP=x$, $PS=y$, $AP=z$, $n=12$, the number of paths and $\angle AOE=\angle COD=\angle DOC=\ldots\ldots\angle NOP=360°\div n=30°$. Then from the right triangles we have $OC=OA\times$

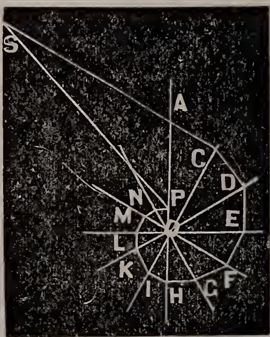


FIG. 93.

$\cos AOC=a\cos\theta$, $OD=OC\cos COD=a\cos^2\theta$, $OE=OD\times\cos DOE=a\cos^3\theta$, $OP=ON\cos NOP=a\cos^n\theta$; $AC=OA\times\sin AOC=a\sin\theta$, $CD=OC\sin COD=a\sin\theta\cos\theta$, $DE=DO\sin DOE=a\sin\theta\cos^2\theta$, $NP=NO\sin NOP=a\sin\theta\cos^{n-1}\theta$. $\therefore z=OA-OP=a(1-\cos^n\theta)$, and $x=a\sin\theta+a\sin\theta\cos\theta+a\sin\theta\cos^2\theta+\ldots\ldots+a\sin\theta\cos^{n-1}\theta=a\sin\theta(1+\cos\theta+\cos^2\theta+\cos^3\theta+\ldots\ldots+\cos^{n-1}\theta)=a\sin\theta(1+\cos^n\theta)\div(1-\cos\theta)=a\cot\frac{1}{2}\theta(1-\cos^n\theta)$. Hence, since $\angle PAS=(90°+\theta)$, we have $y=\sqrt{[x^2+z^2-2xz\times\cos(90°+\theta)]}=a\operatorname{cosec}\frac{1}{2}\theta(1-\cos^n\theta)\times\sqrt{(1+\sin^2\theta)}=\frac{225280}{243}\times\frac{4}{\sqrt{6}-\sqrt{3}}[1-(\frac{3}{4})^6]\times\frac{\sqrt{5}}{2}=3292$ yd., nearly.

NOTE.—This problem was proposed in the *School Visitor*, by Dr. N R. Oliver, Brampton, Ontario. The above elegant solution was given by Prof. E. B. Seitz, and was published in the *School Visitor*, *Vol. 3*, *p. 36*.

I. A fox is 80 rods north of a hound and runs directly east 350 rods before being overtaken. How far will the hound run before catching the fox if he runs towards the fox all the time, and upon a level plain?

Construction.—Let C and A be the position of the hound and fox at the start, P and m corresponding positions of the hound and fox any time during the chase, and P' and n their positions the next instant, B the point where the hound catches the fox and $CPP'B$ the curve described by the hound. Join m and P, and n and P'; they are tangents to the curve at P and P'. Draw Pd and $P'e$ perpendicular to AB, mo perpendicular to $P'n$, and $P'p$ perpendicular to Pd.

II.

1. Let $AC=a=80$ rds.
2. $AB=b=350$ rds.,
3. $Am=x$,
4. $Bd=y$,
5. $Pm=w$,
6. $arc\ CP=s$,
7. curve $CPB=s_1$, and
8. r=ratio of the hound's rate to the fox's. Then we have

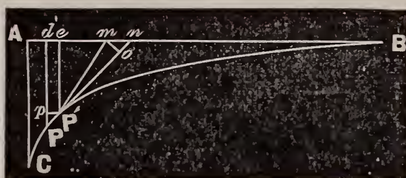


FIG 94.

9. $mn=dx$,
10. $ed=P'p=dy$,
11. $PP'=ds$,
12. $no-PP'=dw$ .... (1), and
13. $s=rx$ .... (2). From (2), we have, by differentiation,
14. $ds=rdx$. Whence,
15. $\frac{dx}{ds}=\frac{1}{r}$. From the similar right triangles PpP' and mon, we have
16. $PP':mn::pP':mo$, or $ds:dx::dy:mo$. Whence,
17. $mo=\frac{dx\times dy}{ds}=\frac{dy}{r}$, since $\frac{dx}{ds}=\frac{1}{r}$. Substituting in (1),
18. $\frac{dy}{r}-ds=dw$, or
19. $dy-r^2dx=rdw$ .... (3). Integrating (3),
20. $y-r^2x=rw+C$ ....(4). But, since when $x=0$, $y=0$, and $w=a$,
21. $0=ra+C$. Whence, $C=-ra$.
22. $\therefore y-r^2x=rw+C=rw-ra$ .... (5). When $x=b$, $y=b$, and $w=0$, and (5) becomes
23. $b-r^2b=-ar$, or $r^2b-ra=b$. Whence,
24. $r^2-\frac{a}{b}r=1$,
25. $r^2-\frac{a}{b}r+\frac{a^2}{4b^2}=1+\frac{a^2}{4b^2}=\frac{a^2+4b^2}{4b^2}$

26. $r-\frac{a}{2b}=\frac{1}{2b}\sqrt{a^2+4b^2}$,
27. $2rb-a=\sqrt{a^2+4b^2}$,
28. $rb=\frac{1}{2}(a+\sqrt{a^2+4b^2})$. But
29. $rb=s_1$, what (1) becomes when $b=x$.
30. $\therefore s_1=\frac{1}{2}(a+\sqrt{a^2+4b^2}=392.2783$ rods, the distance the hound runs to catch the fox.

NOTE.—This solution is substantially the same as the one given by the Late Professor E. B. Seitz, and published in the *School Visitor*, *Vol. IV*, *p. 207*. The path of the hound is known as the "Curve of Pursuit."

I. A ship starts on the equator and travels due north-east at all times; how far has it traveled when its longitude, for the first time, is the same as that of the point of departure?

Let B be the point of the ship's departure, BPN its course, P its position at any time and N its position at the next instant. Then PN is an element of the curve of the ship, which is known as the *Loxodrome*, or *Rhumb* line. Let $\theta=BF=$the longitude of the point P, $\phi=PF=$the corresponding latitude, (x, y, z) the rectangular co-ordinates of P, and $\varphi=\frac{1}{4}\pi=$the constant angle PNQ.

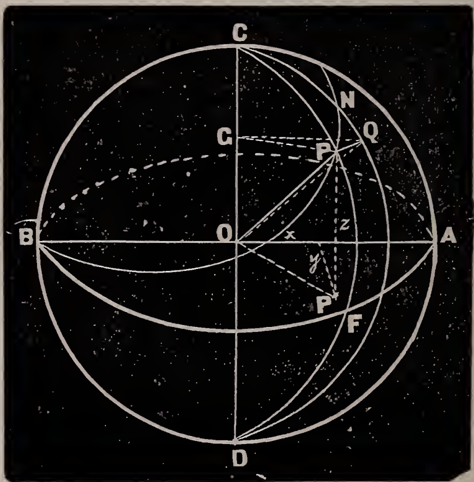


FIG. 95.

Then we have for the equations of the curve, $x=PG\cos\theta$ $=r\cos\phi\cos\theta$ .... (1), $y=PG\sin\theta=r\cos\phi\times\sin\theta$ .... (2), and $z=r\sin\phi$....(3), where r is the radius of the earth. Now an element of a curve of double curvature, referred to rectangular co-ordinates is $\sqrt{(dz^2+dy^2+dx^2)}$. $\therefore PN=ds$ $=\sqrt{(dz^2+dy^2+dx^2}$ ....(4). Differentiating (1), (2), and (3), $dx=-r(\cos\theta\sin\phi d\phi+\cos\phi\sin\theta d\theta)$, $dy=-r(\sin\theta\sin\phi d\phi-\cos\phi\cos\theta d\theta)$, and $dz=r\cos\phi d\phi$. Substituting these values in (4), $ds=r\sqrt{[\cos^2\phi d\phi^2+(\sin\theta\sin\phi d\phi-\cos\phi\cos\theta d\theta)^2+(\cos\theta\sin\phi d\phi+\cos\phi\sin\theta d\theta)^2]}=r\sqrt{(\cos^2\phi d\phi^2+\sin^2\phi d\phi^2+\cos^2\phi d\theta^2)}=r\sqrt{(d\phi^2+\cos^2\phi d\theta^2)}$ .... (4). Now $PQ=GP\times PQ=r\cos\phi d\theta$ and $NQ=rd\phi$. $\frac{PQ}{NQ}=\tan\angle PNQ$

$=\tan\varphi$. $\therefore \dfrac{PQ}{NQ}=\dfrac{\cos\phi d\theta}{d\phi}=\tan\varphi$, or $\cos\phi d\theta=\tan\varphi d\phi$ . . . . (6). Substituting the value of $\cos\phi d\theta$ in (5), $ds=r\sqrt{(d\phi^2+\tan^2\varphi d\phi^2)}$ $=r\sqrt{(1+\tan^2\varphi)}d\phi=\dfrac{r}{\cos\phi}d\phi$. $\therefore s=\dfrac{r}{\cos\varphi}\int_{\phi_2}^{\phi_1}d\phi=$ $\dfrac{r}{\cos\varphi}(\phi_1-\phi_2)$ . . . .(7). By integrating (6), $\theta=\tan\varphi\int\dfrac{d\phi}{\cos\phi}$ $=\tan\varphi\log_e[\tan(\frac{1}{4}\pi+\frac{1}{2}\phi)]$ or $e^{\theta\cot\varphi}=\tan(\frac{1}{4}\pi+\frac{1}{2}\phi)$ . . . (8). Whence, $\phi=2\tan^{-1}(e^{\theta\cot\varphi})-\frac{1}{2}\pi$. When $\theta=2\pi$ and $\varphi=\frac{1}{4}\pi$, $\phi=2\tan^{-1}(e^{2\pi})-\frac{1}{2}\pi=89^\circ\ 47'\ 9''.6=.4988\frac{1}{9}\pi$. $\therefore s=\dfrac{r}{\cos\frac{1}{4}\pi}\times$ $(.4988\frac{1}{9}\pi-0)=r\sqrt{2}(.4988\frac{1}{9}\pi)=2.21615937r=8775.991093$—mi., the distance the ship travels.

The rectangular equations of the Loxodrome are $\sqrt{(x^2+y^2)}\left\{e^{a\tan^{-1}\frac{y}{x}}+e^{-a\tan^{-1}\frac{y}{x}}\right\}=2r$, and $x^2+y^2+z^2=r^2$, where $a=\cot\varphi$.

The last equations are easily obtained from the figure. The first is obtained as follows: From (1) and (2), we find $\theta=\tan^{-1}\dfrac{y}{x}$; also, $x^2+y^2=r^2\cos^2\phi$ or $\cos\phi=\dfrac{1}{r}\sqrt{(x^2+y^2)}$. From (8), we get $e^{\theta\cot\varphi}=\dfrac{\cos\phi}{1+\sin\phi}=\dfrac{\cos\phi}{1+\sqrt{(1-\cos^2\phi)}}$. Whence, $e^{\theta\cot\varphi}+e^{\theta\cot\varphi}\times\sqrt{(1-\cos^2\phi)}=\cos\phi$. Transposing $e^{\theta\cot\varphi}$, squaring, and reducing, we have $\cos\phi(e^{\theta\cot\varphi}+e^{-\theta\cot\varphi})=2$. Substituting the value $\cos\phi$, and θ, we have $\sqrt{(x^2+y^2)}\left\{e^{a\tan^{-1}\frac{y}{x}}+e^{-a\tan^{-1}\frac{y}{x}}\right\}=2r$.

NOTE.—This solution was prepared by the author for problem 1501, *School Visitor*, but it was not published because of its difficult composition.

PROBLEMS.

1. What is the area of a field in the form of a parallelogram, whose length is 160 rods and width 75 rods? *Ans.* 75 A.

2. Find the area of a triangle whose base is 72 rods and altitude 16 rods. *Ans.* 1 A. 2 R. 16 P.

3. Two trees whose heights are 40 and 80 feet respectively, stand on opposite sides of a stream 30 ft. wide. How far does a squirrel leap in jumping from the top of the higher to the top of the lower? *Ans.* 50 feet.

4. How many steps of 3 feet each does a man take in crossing diagonal ly, a square field that contains 20 acres? *Ans.* 440 steps.

5. Find the cost of paving a court 150 feet square; a walk 10 feet around the whole being paved with flagstones at 54 cents a square yard and the rest at $31\frac{1}{2}$ cents a square yard? *Ans.* \$939.40.

6. What is the area of a triangle, the three sides of which are respectively 180 feet, 150 feet, and 80 feet? *Ans* 5935.85 sq. ft.

7. What is the area of a trapezium, the diagonal of which is 110 feet, and the perpendiculars to the diagonals are 40 feet and 60 feet respectively? *Ans.* 5500 sq. ft.

8. At 30 cents a bushel, find the cost of a box of oats, the box being 8 feet long, 4 feet wide and 4 feet deep. *Ans.* \$30.85$\frac{1}{2}$.

9. Two trees stand on opposite sides of a stream 40 feet wide. The height of one tree is to the width of the stream as 8 is to 4, and the width of the stream is to the height of the other as 4 is to 5. What is the distance between their tops? *Ans.* 50 feet.

10. How many miles of furrow 15 in., wide, is turned in plowing a rectangular field whose width is 30 rods and length 10 rods less than its diagoal? *Ans.* $49\frac{1}{2}$ mi.

11. The sides of a certain trapezium measures 10, 12, 14, and 16 rods respectively, and the diagonal, which forms a triangle with the first two sides, is 18 rods; what is the area? *Ans.* 163.796 sq. rds.

12. Three circles, each 40 rods in diameter, touch each other externally; what is the area of the space inclosed between the circles? *Ans.* 64.5 sq. rds.

13. How many square inches in one face of a cube which contains 2571353 cubic inches? *Ans.* 18769 sq. in.

14. Four ladies bought a ball of thread 3 inches in diameter; what portion of the diameter must each wind off to heve equal shares of the thread?

Ans. First, .2743191 in.; Second, .3445792 in.; Third, .4912292 in.; Fourth, 1.8898815 in.

15. A gentleman proposed to plant a vineyard of 10 A. If he places the vines 6 feet apart; how many more can he plant by setting them in the quincunx order than in the square order, allowing the plat to lie in the form of a square, and no vine to be set nearer its edge than 1 foot in either case? *Ans.* 1870.

16. Find the volume generated by the revolution of a circle about a tangent. *Ans.* $2\pi^2 R^2$.

17. How many feet in a board 14 feet long and 16 inches wide at one end and 10 inches at the other, and 3 inches thick? *Ans.* $45\frac{1}{2}$ feet.

18. If I saw through $\frac{1}{4}$ of the diameter of a round log, what portion of the cut is made? *Ans.* .196.

19. What is the surface of the largest cube that can be cut from a sphere which contains 14137.2 cu. ft.? *Ans*. 1800 sq. ft.

20. Two boys are flying a kite. The string is 720 feet long. One boy who stood directly under the kite, was 56 feet from the other boy who held the string; how high was the kite? *Ans*. 717.8+feet.

21. How many pounds of wheat in a cylindrical sack whose diameter is $1\frac{1}{2}$ feet, and whose length is $1\frac{3}{4}$ yards? ($\pi=3.1416$) *Ans*. 447.31 feet.

22. How large a square can be cut from a circle 50 inches in diameter? *Ans*. 35.3553391 in.

23. How many bbl. in a tank in the form of the frustum of a pyramid, 5 feet deep, 10 feet square at the bottom and 9 feet square at the top? *Ans*. 107.26 bbl.

24. From a circular farm of 270 acres, a father gives to his sons equal circular farms, touching each other and the boundary of the farm. He takes for himself a circular portion in the center, equal in area to a son's part, and reserves the vacant tracts around his part for pasture lands and gives each son one of the equal spaces left along the boundary. Required the number of sons and the amount of pasture land each has. *Ans*. 6 sons; 8.46079 A.

25. At each angle of a triangle being on a level plain and having sides respectively 40, 50, and 60 feet, stands a tower whose height equals the sum of the two sides including the angle. Required the length of a ladder to reach the top of each tower without moving at the base. *Ans*. 116.680316+ft.

26. If the door of a room is 4 feet wide, and is opened to the angle of 90 degrees, through what distance has the outer edge of the door passed? *Ans*. 6.2832 feet.

27. A tinner makes two similar rectangular oil cans whose inside dimensions are as 3, 7, and 11. The first hold 8 gallons and the second being larger requires 4 times as much tin as the other. What are the dimensions of the smaller and the contents of the larger?

Ans. { Dimensions of smaller 6, 14, and 22 inches.
Capacity of larger 64 gallons.

28. An 8-inch globe is covered with gilt at 8 cents per square inch; find the cost. *Ans*. $16.08.

29. A hollow cylinder 6 feet long, whose inner diameter is 1 inch and outer diameter two inches, is transformed into a hollow sphere whose outer diameter is twice its inner diameter; find outer diameter. *Ans*. 3.59 in.

30. A circular field is 360 rods in circumference; what is the diagonal of a square field containing the same area? *Ans*. 20.3 rods.

31. What is the volume of a cylinder, whose length is 9 feet and the circumference of whose base is 6 feet? *Ans*. 25.78 cu. ft.

32. How many acres in a square field, the diagonal being 80 rods? *Ans*. 20 acres.

33. How many cubical blocks, each edge of which is $\frac{1}{3}$ of a foot, will fill a box 8 feet long, 4 feet wide, and 2 feet thick. *Ans*. 1728 blocks.

34. From one corner of a rectangular pyramid 6 by 8 feet, it is 19 feet to the apex; find the dimentions of a rectangular solid whose dimensions are as 2, 3, and 4, that may be equivalent in volume. *Ans*. 4, 9, and 8 feet.

35.* A solid metal ball, 4 inches radius, weighs 8 lbs.; what is the thickness of spherical shell of the same metal weighing $7\frac{5}{8}$ lb., the external diameter of which is 10 inches? *Ans*. 1 inch.

36. What is the difference between 25 feet square and 25 square feet? *Ans*. 600 sq. ft.

37.* Find the greatest number of trees that can be planted on a lot 11 rods square, no two trees being nearer each other than one rod?

Ans. 152 trees.

38.* A straight line 200 feet long, drawn from one point in the outer edge of a circular race track to another point in the same, just touches the inner edge of the track. Find the area of the track and its width.

Ans. Area, $\pi a^2 = 4000\pi$ sq. ft.; width, *indeterminate*

39. The perimeter of a certain field in the form of an equilateral triangle is 360 rods; what is the area of the field? *Ans.* 543.552 sq. rd.

40. A room is 18 feet long, 16 feet wide, and 10 feet high. What length of rope will reach from one upper corner to the opposite upper corner and touch the floor? *Ans.* 35.3 ft.

41. How many bushels of wheat in a box whose length is twice its width, and whose width is 4 times its height; diagonal being 9 feet?

Ans. 25 bu., *nearly.*

42. Find the area of a circular ring whose breadth is 2 inches and inside diameter 9 inches. *Ans.* 69.1152 sq. in.

43.* A round stick of timber 12 feet long, 8 inches in diameter at one end and 16 inches at the other, is rolled along till the larger end describes a complete circle. Required the circumference of the circle.

Ans. 150.83 feet.

44. A fly traveled by the shortest possible route from the lower corner to the opposite upper corner of a room 18 feet long, 12 feet wide and 10 feet high. Find the distance it traveled. *Ans.* 28.42534 feet.

45.* From the middle of one side and through the axis perpendicularly of a right triangular prism, sides 12 inches, I cut a hole 4 inches square. Find the volume removed. *Ans.* 138.564064 cu. in.

46.* Two isosceles triangles have equal areas and perimeters. The base of one is 24 feet, and one of the equal sides of the other is 29 feet. The area of both is 10 times the area of a triangle whose sides are 13, 14, and 15 feet. Find the perimeters and altitudes.

Ans. Perimeters, 98 feet; altitudes 35 and 21 feet.

47. A grocer at one straight cut took off a segment of a cheese which had $\frac{1}{4}$ of the circumference, and weighed 3 pounds; what did the whole weigh? *Ans.* 33.023 lb.

48.* A twelve inch ball is in a corner where walls and floor are at right angles; what must be the diameter of another ball which can touch that ball while both touch the same floor and the same walls?

Ans. 3.2154 in. or 44.7846 in.

49. What will it cost to paint a church steeple, the base of which is an octagon, 6 feet on each side, and whose slant height is 80 feet, at 30 cents per square yard? *Ans.* \$64.

50. A tree 48 feet high breaks off; the top strikes the level ground 24 feet from the bottom of the tree; find the height of the stump. *Ans.* 18 feet.

51. How many acres in a square field whose diagonal is $5\frac{1}{4}$ rods longer than one of its sides? *Ans.* 160.6446 sq. rd.

52.* Three poles of equal length are erected on a plane so that their tops meet, while their bases are 90 feet apart, and distance from the point where the poles meet to the center of the triangle below is 65 feet. What is the length of the poles? *Ans.* 83.23 feet.

53. A field contains 200 acres and is 5 times as long as wide. What will it cost to fence it, at a dollar per rod? *Ans.* \$960.

54.* What is the greatest number of plants that can be set on a circular piece of ground 100 feet in diameter, no two plants to be nearer each other than 2 feet and none nearer the circumference than 1 foot? *Ans.* 2173.

55. The axes of an ellipse are 100 inches and 60 inches; what is the difference in area between the ellipse and a circle having a diameter equal to the conjugate axis? *Ans.* $600\pi = 1884.96$ sq. in.

56. Find the diameter of a circle of which the altitude of its greatest inscribed triangle is 25 feet. *Ans.* $33\frac{3}{4}$ feet.

57. If we cut from a cubical block enough to make each dimension 1 inch shorter, it will lose 1657 cubic inches, what are the dimensions?

58. Show that the area of a rhombus is one-half the rectangle formed by its diagonals. *Noble Co. Ex. Test.*

59. The length and breadth of a rectangular field are in the ratio of 4 to 3. How many acres in the field, if the diagonal is 100 rods?

60. A spherical vessel 30 inches in diameter contains in depth, 1 foot of water; how many gallons will it take to fill it? *Holmes Co. Ex. Test.*
Ans. 39 gallons.

61. A field is 40 rods by 80 rods. How long a line from the middle of one end will cut off $7\frac{1}{2}$ acres? *Ans.* 80.6 rd., *nearly.*

62. A ladder 20 feet long leans against a perpendicular wall at an angle of 30°. How far is its middle point from the bottom of the wall?
Ans. 10 feet.

63. Four towers, A 125 feet high, B 75 feet, C 160 feet, and D 65 feet, stand on the same plane. B due south and 40 rods from A; C east of B and D south of C. The distance from A to C plus the distance from C to B is half a mile, and the distance from D to B is $82\frac{1}{2}$ yd. farther than the distance from C to D. What length of line is required to connect the tops of A and D? *Ans.* 240+rds.

64. Find the volume of the largest square pyramid that can be cut from a cone 9 feet in diameter and 20 feet high? *Ans.* 270 cu. ft.

65. A rectangular lawn 60 yd. long and 40 yd. wide has a walk 6 ft. wide around it and paths of the same width through it, joining the points of the opposite sides. Find in square yards the area of one of the four plats inclosed by paths. *Ans.* 459 sq. yd.

66. Which has the greater surface, a cube whose volume is 13.824 cu. ft., or a rectangular solid of equal volume whose length is twice its width, and its width twice its height? *Ans.* Cube, 424 sq. ft., more.

67. The volume of a rectangular tin can is 3 cu. ft. 1053 cu. in.; its dimensions are in the proportion of 11, 7, and 3. Find the area of tin in the can. *Ans.* $16\frac{3}{8}$ sq. ft.

68. A conical well has a bottom diameter of 28 ft. 3 in., top diameter 56 ft. 6 in., and depth 23 ft. 1.2 in. Find its capacity in barrels.
Ans. 8023 bbl.

69. A cylindrical vessel 1 foot deep and 8 inches in diameter was $\frac{13}{16}$ full of water; after a ball was dropped into the vessel it was full. Find the diameter of the ball. *Ans.* 6 inches.

70. Two logs whose diameters are 6 feet lie side by side. What is the diameter of a third log placed in the crevice on top of the two, if the pile is 9 feet high?. *Ans.* 4 ft.

71. Circles 6 and 10 feet in diameter touch each other; if perpendiculars from the center are let fall to the line tangent to both circles, how far apart will they be? *Ans.* 7.756 ft.

72. What are the linear dimensions of a rectangular box whose capacity is 65910 cubic feet; the length, breadth, and depth being to each other as 5, 3, and 2? *Ans.* 65, 39, and 26 ft.

73. The perimeter of a piece of land in the form of an equilateral triangle is 624 rods; what is the area? *Ans.* 117 A. 13 31 P.

74. Four logs 4 feet in diameter lay side by side and touch each other; on these and in the crevices lay three logs 3 feet in diameter; on these three and in the crevices lay two logs 2 feet in diameter; what is the diameter of a log that will lay on the top of the pile touching each of the logs 2 feet in diameter and the middle one of the logs 3 feet in diameter?

Ans. ———

75. What will it cost to gild a segment of a sphere whose diameter is 6 inches; the altitude of the segment being 2 inches, at 5 ¢ per square inch?

Ans ———.

76. A grocer cut off the segment of a cheese, and found it took $\frac{1}{6}$ of the circumference. What is the weight of the whole cheese, if the segment weighed $1\frac{1}{2}$ lbs? *Ans.* 52.0228+lbs.

77. Two ladders are standing in the street 20 feet apart. They are inclined equally toward each other at the top, forming an angle of 45°. Find, by arithmetic, the length of the ladders? *Ans.* 26.13 ft.

Union Co. Ex. List.

78. Two trees stand on opposite sides of a stream 120 feet wide; the height of one tree is to the width of the stream as 5 is to 4, and the width of the stream is to the height of the other as 4 is to 5; what is the distance between their tops? *Ans.* 136.95 ft.

79. How many gallons of water will fill a circular cistern 6 feet deep and 4 feet in diameter? *Ans.* 564.0162 gal.

80. A cube of silver, whose diagonal is 6 inches, was evenly plated with gold; if 4 cubic inches of gold were used, how thick was the plating?

Ans. $\frac{1}{648}$. in.

81. Required the distance between the lower corner and the opposite upper corner of a room 60 feet long, 32 feet wide, and 51 feet high?

Ans. 85 ft.

82. How deep must be a rectangular box whose base inside is 4 inches by 4 inches to hold a quart, dry measure? *Ans.* 4.2 cu. in.

83. A fly is in the center of the floor of a room 30 feet long, 20 feet wide, and 12 feet high. How far will it travel by the shortest path to one of the upper corners of the ceiling? *Ans.* 24.67+ft.

84. A corn crib 25 feet long holds 125 bushels. How many bushels will one of like shape and 35 feet long hold?

85. Let a cube be inscribed in a sphere, a second sphere in this cube, a second cube in this sphere, and so on; find the diameter of the 7th sphere, if that of the first is 27 inches. (2). What is the volume of all the spheres so inscribed including the first? *Ans.*————.

86. The area of a rectangular building lot is 720 sq. ft.; its sides are as 4 to 5; what will it cost to excavate the earth 7 feet deep at 36¢ per cubic yard? *Ans.* $76.20.

87. A owns $\frac{1}{3}$ and B the remainder of a field 60 rods long and 30 rods wide at one end and 20 rods wide at the other end, both ends being parallel to the same side of the field They propose to lay out through it, parallel with the ends, a road one rod wide leaving A's $\frac{1}{3}$ of the remainder at the wide end and B's $\frac{2}{3}$ at the narrow end of the field. Required the location and area of the road. *Ans* ———.

88. The diameter of a circular field is 240 rods. How much grass will be left after 7 horses have eaten all they can reach, the ropes which are allowed them being of equal lengths and attached to posts so located that each can touch his neighbor's territory and none can reach beyond the boundary of the field? *Ans.* 62.831853 A.

89. What is the diameter of a circle inclosing three equal tangent circles, if the area inclosed by the three equal circles is 1 acre? *Ans.* ———.

90. What is the diameter of a circle inclosing four equal tangent circles each being tangent to the the required circle, if the area inclosed by the four equal circles is 1 acre? *Ans.* $R=4\sqrt{[5(4-\pi)]}\sqrt{\ }2+1)\div(4-\pi)$.

91. What is the greatest number of stakes that can be driven one foot apart on a rectangular lot whose length is 30 feet and width 20 feet?

Ans.————.

92. What is the greatest number of inch balls that can be put in a box 15 inches long, 9 inches wide, and 6 inches high? *Ans* ————.

93. A conical vessel 6 inches in diameter and 10 inches deep is full of water. A heavy ball 8 inches in diameter, is put into the vessel; how much water will flow out? *Ans* ————.

94. How far above the surface of the earth would a person have to ascend in order that $\frac{1}{3}$ of its surface would be visible? *Ans.* 8000 mi.

95. Where must a frustum of a cone be sawed in two parts, to have equal solidities, if the frustum is 10 feet long, 2 feet in diameter at one end, and 6 feet at the other? *Ans* ———.

96. At the three corners of a rectangular field 50 feet long and 40 feet wide, stands three trees whose heights are 60, 80, and 70 feet. Locate the point where a ladder must be placed so that without moving it at the base it will touch the tops of the three trees, and find the length of the ladder. What must be the height of a tree at the fourth corner so that the same ladder will reach the top, the foot of the ladder not being moved?

Ans. ————

97. A horse is tied to a corner of a barn 50 feet long and 30 feet wide; what is the area of the surface over which the horse can graze, if the rope is 80 feet long? *Ans.* ————.

98 How many cubic feet in a stone 32 feet high, whose lower base is a rectangle, 10 feet by 4 feet and the upper base 8 feet by $1\frac{1}{2}$ feet?

Ans. $805\frac{1}{3}$ cu. ft.

99. To what height above the ground would a platform, 10 feet by 6 feet, have to be elevated so that 720 sq. ft. of surface would be invisible to a man standing at the center of the platform, the man being 5 feet high?

Ans ————.

100. Required the side of the least equilateral triangle that will circumscribe seven circles, each 20 inches in diameter. *Ans.* 89.28203 in.

101. Required the sides of the least right triangle that will circumscribe seven circles each 20 inches in diameter. *Ans* 123.9320 in. and 107.3205 in.

102. How long a ladder will be required to reach a window 40 feet from the ground, if the distance of the foot of the ladder from the wall is $\frac{3}{5}$ of the length of the ladder. *Ans.* 50 ft.

103. A circular park is crossed by a straight path cutting off $\frac{1}{4}$ of the circumference; the part cut off contains 10 acres Find the diameter of the park. *Ans.* 150 rd., *nearly*

104. Find the length of the minute-hand of a clock, whose extreme point moves 5 ft. 5.9736 in., in 1 da. 18 hr.? *Ans.* 3 in.

105. A, B, and C, own a triangular tract of land. Their houses are located at the vertices of the triangle; where must they locate a well to be used in common so that the distance from the houses to the well will be the same, the distance from A to B being 120 rods, from B to C 90 rods and A to C 80 rods. *Ans.* ————.

106. A horse is tethered from one corner of an equilateral triangular building whose sides are 100 feet, by a rope 175 feet long. Over what area can he graze? *Ans.* 90021.109181 sq. ft.

107. Find the area of the triangle formed by joining the centers of the squares constructed on the sides of an equilateral triangle, whose sides are 20 feet? *Ans.* ————.

GEOMETRY

1. ***Geometry*** is the science that treats of position and extension.

2. ***Pure Geometry*** $\left\{\begin{array}{l}\text{1. Plane.}\\ \text{2. Solid.}\end{array}\right.$

Problem.—*To bisect a given triangle by a line drawn from a random point in one of its sides.*

Demonstration.—Let ABC be the given triangle, D a random point in the side BC, and E the middle point of BC. Join A and D, A and E. Draw EF parallel to AD. Draw DF. Then DF bisects the triangle ABC. For the triangle ABE is equivalent to the triangle AEC (?). The triangle AED is equivalent to the triangle ADE (?). Hence, $ABDF$ is equivalent to ABE (?) and, therefore, DF bisects the triangle ABC. *Q. E. D.*

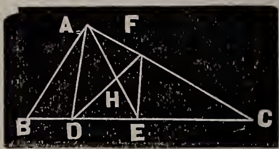


FIG. 1.

Proposition.—*The square described upon the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.*

I. *Demonstration.*—Let CFD be any right triangle, right angled at F and let AC, CP, and DM be the squares described upon its sides. Then the square AC is equal to the sum of the squares CP and DM. Through F, draw QF perpendicular to AB and produce it to meet OP produced, in G; also produce BC to meet OP in I and AD to meet OP produced, in R. Draw GH parallel to PD, and BT parallel to CF. Draw AE. Now the triangles COI and DFC are equal (?). Hence, $CI=CD=CB$, and therefore the square CP=the parallelogram CG (?)=the parallelogram BF (?)=the rectangle BK (?). In like manner, the square DM can be proved equal to the rectangle AK. Hence, the square AC=the square $CP+$ the square DM. *Q. E. D.*

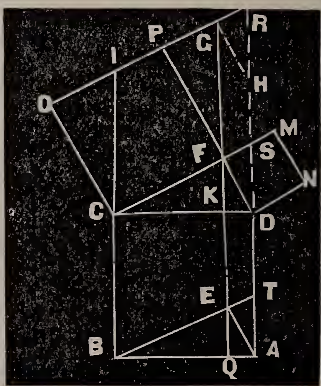


FIG. 2.

II. *Demonstration.*—Let EDC be any right triangle, right angled at D. On the sides DE and DC construct the squares $EDHG$ and $DCBM$ respectively. Produce GE and BC until they meet in F, forming the square $FBAG$. On EC, the hypotenuse, construct the square $ECKI$. Then the square $ECKI$ is equal to the sum of the squares $EDHG$ and $DCBM$. For, the square $GFBA$ is equal to $GEDH+DCBM+2\ EDCF$ $(=4ECF)$. The square $GFBA$ is also equal to the square $ECKI+4ECF$. Hence, $ECKI+4ECF= GEDH+DCBM+4ECF$ (?). Whence, $ECKI =GEDH+DCBM$. *Q. E. D.*

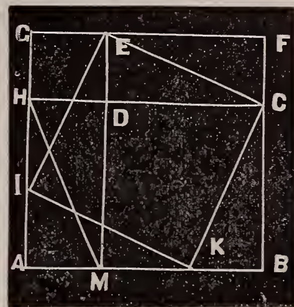


FIG. .3

Proposition.—*In any triangle, each angle formed by joining the feet of the perpendiculars is bisected by the perpendicular from the opposite vertex.*

Demonstration.—Let ABC be any triangle and AD, BE, and CF the three perpendiculars. Join D and E, D and F, and E, and F.

In the right triangles AEB and AFC, the angle BAC is common to both. Therefore, they are similar. Hence, AB:AC $=AE$:AF. Now the triangles BAC and FAE have the angle FAE common and the including sides proportional. Therefore, they are similar, and the angle AFE=the angle ACB. In a similar manner we may prove that the angle DFB=the angle ACB; the angle AFE=the angle DFB. From this it follows that the angle CAF —the angle EFA=the angle CFB— the angle DFB. Hence, angle EFC=angle CFD and the angle EFD is bisected by the perpendicular CF. In a similar manner,it can be proved that AD bisects the angle FDE and EB bisects the angle ABC. *Q. E. D.*

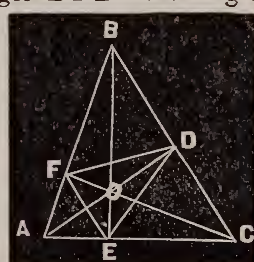


FIG. 4.

Problem.—*From a given point in an arc less than a semicircumference, draw a chord of the circle which will be bisected by the chord of the given arc.*

Demonstration.—Let $ABDC$ be the given circle, AB the given arc, AB the chord of the arc, and P any point of the arc

APC. Draw the diameter POC and on the radius PO as a diameter describe the circle PEO. Then through the points E, and G, of intersection draw the chords PD and PF respectively, and they will be bisected at the points E and G. For draw DC and OE. Then the triangles PEO and PDC are right triangles(?) and are also similar (?). Since PEO and PDC are similar, the line OE is parallel to DC, and since O is the middle point of PC,E is the middle point of PD(?). In like manner, G is the middle point of PF. Q. E. F.

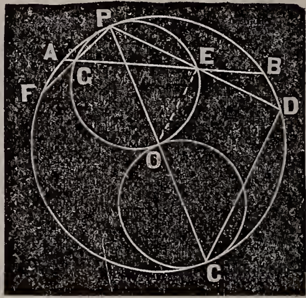


FIG. 5.

Discussion.—There are, in general, two solutions. When arc AB is diminished until B coincides with A, there is no solution. When AB is a semi-circumference, there is one solution and the chord is the diameter POC.

Proposition.—*If two equal straight lines intersect each other anywhere at right angles, the quadrilateral formed by joining their extremities is equivalent to half the square on either straight line.*

Demonstration.—Let AB and CD be two equal straight lines intersecting each other at right angles at E. Join their extremities, forming the quadrilateral $ACBD$. Then $ACBD$ is equivalent to half the square of AB or CD. For the area of the triangle ACB equals $\frac{1}{2}(AB \times CD)$ and the area of the triangle ADB equals $\frac{1}{2}(AB\times ED)$. Hence, the area of $ACBD=\frac{1}{2}(AB\times ED)+\frac{1}{2}(AB\times EC)=\frac{1}{2}AB(ED+EC)=\frac{1}{2}(AB\times CD)$. But CD equals AB, by hypothesis. Hence, $ACBD=\frac{1}{2}AB^2$. Q. E. D.

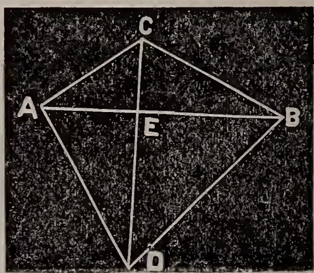


FIG. 6.

A PROBLEM IN MODERN GEOMETRY.

An equilateral hyperbola passes through the middle points D, E, and F of the sides BC, AC, and AB of the triangle ABC, and cutting those sides in order in α, β, and γ. Show that the lines $A\alpha$, $B\beta$, and $C\gamma$ intersect in a point the locus of which is the circumscribing circle of the triangle ABC.

Solution.—The equation to any conic is $u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\alpha\gamma+2w'\alpha\beta=0$ .... (1). D is $(0, \frac{1}{2}a \sin C, \frac{1}{2}a \sin B)$; E, $(\frac{1}{2}b \sin C,\ 0, \frac{1}{2}b \sin A)$; F, $(\frac{1}{2}c \sin B, \frac{1}{2}c \sin A, 0)$. These

points being on (1), we should have $c^2v+b^2w+2bcu'=0$ ... (2), $c^2u+a^2w+2acv'=0$ ..... (3), $b^2u+a^2v+2abw'=0$ ....(4).

Whence $u=\frac{a}{bc}(au'-bv'-cw')$ .... (5), $v=\frac{b}{ac}(bv'-cw'-au')$ .. ...(6), $w=\frac{c}{ab}(cw'-au'-bv')$ ....(7). Substituting in the condition $u+v+w-2u'\cos A-2v'\cos B-2w'\cos C=0$ .... (8) that (1) is an equilateral hyperbola,

$$\frac{a^2(au'-bv'-cw')+b^2(bv'-cw'-au')+c^2(cw'-au'-bv')}{abc}$$

$-2u'\cos A-2v'\cos B-2w'\cos C=0$..... (9). Clearing of fractions and noticing that $2abc\cos A=a(b^2+c^2-a^2)$ ..... (10), $2abc\cos B=b(a^2+c^2-b^2)$ ....(11), $2abc\cos C=c(a^2+b^2-c^2)$ ..... (12), and reducing, $u'\cos A+v'\cos B+w'\cos C=0$....(13). Substituting (5), (6), and (7) in (1) an clearing of fractions, $a^2(au'-bv'-cw')\alpha^2+b^2(bv'-cw'-au')\beta^2+c^2(cw'-au'-bv')$ $+\gamma^2+2u'abc\beta\gamma+2v'abc\alpha\gamma+2w'abc\alpha\beta=0$ ... (14). Where this cuts BC, $\alpha=0$, and (14) gives $b^2(bv'-cw'-au')\frac{\beta^2}{\gamma^2}+2abcu'\frac{\beta}{\gamma}$ $=-c^2(cw'-au'-bv')$ ... (15), whence for the point α; $\alpha_1=0$, $\beta_1=\frac{c}{b}\frac{cu'-bv'-au'}{-cu'+bv'-au'}\gamma_1$ ....(16). By symmetry, for the point. β, $\alpha_2=\frac{c}{a}\frac{cw'-au'-bv'}{-cw'+au'-bv'}\gamma_2$, $\beta_2=0$ .... (17). The equation to $A\alpha$ is found to be $b(-cw'+bv'-aw')\beta-(cw'-bv'-au')\gamma=0$. ... (18); to $B\beta$, $a(-cw'+au'-bv')\alpha-c(cw'-au'-bv')\gamma=0$. ....(19); and to $C\gamma$, $b(-au'+bv'-cw')\beta$ $-a(au'-bv'-cw')\alpha=0$ .... (20), any two of which meet in

$$\left\{\begin{array}{l} \alpha'=\dfrac{bc(-cw'+bv'-au')(cw'-au'-bv')}{D_1} \\ \beta'=\dfrac{ac(cw'-bv'-au')(-cw'+au'-bv')}{D_1}, \\ \gamma'=\dfrac{ab(-cw'+bv'-au')(-cw'+au'-bv')}{D_1} \end{array}\right. \quad \ldots\ldots (21).$$

The circumscribing circle is $a\beta\gamma+b\alpha\gamma+c\alpha\beta=0$...... (22), which is satisfied by (21) on condition (13), proving the proposition.

NOTE.—This problem was solved by Professor William Hoover, A. M., Ph. D., Professor of Mathematics and Astronomy in the Ohio University, Athens, Ohio, who is one the leading mathematicians in the United States, and whose biography follows.

BIOGRAPHY.

PROF. WILLIAM HOOVER, A. M., PH. D.

Professor Hoover was born in the village of Smithville, Wayne county, Ohio, October 17, 1850, and is the oldest of a family of seven children. Both parents are living in the village where he was born, still enjoying good health.

Up to the age of fifteen he attended the public schools, and for two or three years after, a local academy. Owing to needy circumstances he was obliged to work for his living quite early, and almost permanently closed attendance at any kind of school at eighteen years of age, sometime before which, going into a store in the county seat, as clerk. Nothing could have been farther from his taste than this work, having been thoroughly in love with study and books long before. After spending two or three years in this way, he went to teaching, about the year 1869, and he has been regularly engaged in his favorite profession to the present day.

He attended Wittenberg College and Oberlin College one term each, a thing having very little bearing on his education. He studied no mathematics at either place, excepting a little descriptive astronomy at the latter.

After teaching three winters of country school, with indifferent success, he was chosen, in 1871, a teacher in the Bellefontaine, Ohio, High School, serving one year, when he was given a place in the public schools of South Bend, Ind. Remaining there two years, he was invited to return to Bellefontaine as superintendent of schools. He afterwards served in the same capacity in Wapakoneta, O., two years, and as principal of the second district school of Dayton, O. In 1883, he was elected professor of mathematics and astronomy in the Ohio University, Athens, Ohio, where he is still in service.

Through all his career of teaching, Professor Hoover has been an incessant student, devoting himself largely to original investigations in mathematics. Although his pretentions in other lines are very modest, he is eminently proficient in literature, language, and history. Before going into college work he had collected a good library. He is indebted to no one for any attainments made in the more advanced of these lines, but by indefatigable energy and perseverance he has made himself the cultured, classic, and renowned scholar he is.

He has always been a thorough teacher, aiming to lead pupils to a mastery of subjects under consideration. His habits of mind and preparation for the work show him specially adapted to his present position, where he has met great success. He studies methods of teaching mathematics, which in the higher parts is supposed to be dry and uninteresting. He sets the example of enthusiasm as a teacher, and rarely fails to impress upon the minds of his students the immense and varied applications of mathematics. He is kind and patient in the class-room and is held in the highest esteem by his students. He is ever ready to aid the patient student inquiring after truth. It seems to be a characteristic of eminent mathematicians that they desire to help others to the same heights to which they themselves have climbed. This was true of Prof. Seitz; it is true of Dr. Martin; and it is true of Prof. Hoover.

In 1879, Wooster University conferred upon Prof. Hoover the degree of Master of Arts, and, in 1886, the degree of Doctor of Philosophy *cum laude*, he submitting a thesis on Cometary Perturbations. In 1889, he was elected a member of the London Mathematical Society and is the only man in his state enjoying this honor. In 1890, he was elected a member of the New York Mathematical Society. He has been a member of the Asso-

Yours Truly
William Hoover

ciation for the Advancement of Science for several years. Papers accepted by the association at the meetings at Cleveland, Ohio, and at Washington, D. C., have been presented on "The Preliminary Orbit of the Ninth Comet of 1886," and "On the Mean Logarithmic Distance of Pairs of Points in Two Intersecting Lines." He is in charge of the correspondence work in mathematics in the Chautauqua College of Liberal Arts and of the mathematical classes in the summer school at Lake Chautauqua, the principal of which is the distinguished Dr. William R. Harper, president of the new Chicago University. The selection of Professor Hoover for this latter position is of the greatest credit, as his work is brought into comparison with some of the best done anywhere.

He is a critical reader and student of the best American and European writers, and besides, is a frequent contributor to various mathematical journals, the principal of which are *School Visitor*, *Mathematical Messenger*, *Mathematical Magazine*, *Mathematical Visitor*, *Analyst*, *Annals of Mathematics*, and *Educational Times*, of London, England.

His style is concise and his aim is elegance in form of expression of mathematical thought. While greatly interested in the various branches of pure mathematics. he is specially interested in the applications to the advanced departments of Astronomy, Mechanics, and the Physical Sciences —such as Heat, Optics, Electricity, and Magnetism. The "electives" offered in the advanced work for students in his University are among the best mathematics pursued anywhere in this country.

He is an active member of the Presbyterian church and greatly interested in every branch of church work. He has been an elder for a number of years and was chosen a delegate to the General Assembly meeting at Portland, Oregon, in May, 1892, serving the church in this capacity with fidelity and intelligence. In this biography of Professor Hoover, there is a valuable lesson to be learned. It is this: Energy and perseverance will bring a sure reward to earnest effort. We see how the clerk in a county seat store, in embarrassing circumstances and unknown to the world of thinkers, became the well known Professor of Mathematics and Astronomy in one of the leading Institutions of learning in the State of Ohio. "Not to know him argues yourself unknown."

THE NINE-POINT CIRCLE.

Proposition.—*If a circle be described about the pedal triangle of any triangle, it will pass through the middle points of the lines drawn from the orthocenter to the vertices of the triangle, and through the middle points of the sides of the triangle, in all, through nine points.* (2) *The center of the nine-point circle is the middle point of the line joining the orthocenter and the center of the circumscribing circle of the triangle.* (3) *The radius of the nine-point circle is half the radius of the circumscribing circle of the triangle.* (4) *The centroid of the triangle also lies on the line joining the orthocenter and the center of the circumscribing circle of the triangle, and divides it in the ratio of 2:1.* (5) *The sides of the pedal triangle intersect the sides of the given triangle in the radical axis of the circumscribing and nine-point circles.* (6) *The nine-point circ' touches the inscribed and escribed circles of the triangle.*

The Pedal Triangle is a triangle formed by joining the feet of the perpendiculars drawn from the vertices of a triangle to the opposite sides.

The Orthocenter is the point of intersection of these perpendiculars.

Medial Lines, or ***Medians***, are lines drawn from the vertices of a triangle to the middle point of the opposite sides.

The Centroid is the point of intersection of the medians.

The Radical Axis of two circles is the locus of the points whose powers with respect to the two circles are equal.

Demonstration.—Let ABC be any triangle, AD, BF, and CE the perpendiculars from the vertices to the opposite sides of the triangle. O is the orthocenter. Join the points F, E, and D. Then FED is the pedal triangle. About this triangle, describe the circle $FEHDK$. It will then pass through the middle points L, N, and R of the lines, OA, OB, and OC, and the middle points H, G, and K of the sides AB, BC, and AC, in all, through nine points.

Since the angles AFO and AEO of the quadrilateral are both right angles a circle may be described about it. For the same reason circles may be described about the quadrilaterals $EBDO$ and $ODCF$. Draw the lines FR and RG. Now the angles FRE and FDE are equal, being measured by half the same arc FE. But FDE equals $2EDL$, because AD bisects the angle EDF. $\therefore$ FRO equals $2FDL$. Both being measured by the same arc OF, and FRO being two times FDL, FRO is an angle at the center; therefore, since OC is the diameter of the circle circumscribing $FODC$, R is the middle point of OC. In like manner it may be proved that OB and OA are bisected in the points N and L respectively. Draw the line RG. The angles RGC and RGB are equal to two right angles. Also the angles RGB and RED are equal to two right angles, because they are opposite angles of a quadrilateral inscribed in a circle. Therefore RGC is equal to RED. But RED is equal to OBD, because both are measured by half the arc OD. $\therefore$ The angle RGC equals the angle OBD, and consequently the line RG is parallel to the line OB. But, since RG bisects OC in R and is parallel to OB, it bisects BC in G. In like manner, it may be shown that AB and AC are bisected by the nine-point circle in the points H and K respectively. Hence, the circle passes, in all, through nine points. *Q. E. D.*

(2.) Draw the perpendiculars GP, KP, and HP from the middle points of the sides of the triangle. They all meet in a common point P which is the center of the circumscribing circle of the triangle. With P as a center and radius equal to PB,

describe the circumscribing circle. Draw the perpendiculars SY SJ, and SZ to the middle points of the chords FK, EH, and

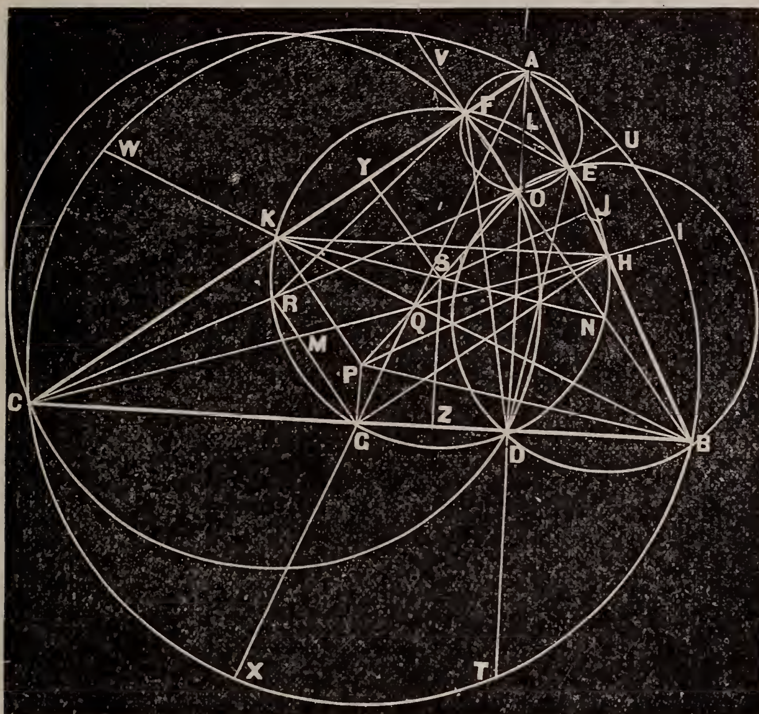


FIG. 6.

DG. These all meet in the same point S, which is the center of the nine-point circle. In the trapezoid $PHEO$, since SJ bisects EH in J and is parallel to PH, it bisects OP in S. Hence, the center of the nine-point circle is the middle point of the line joining the orthocenter and center of the circumscribing circle. $Q.\ E.\ D.$

(3.) Draw the lines KN and PB. Since the angle KFN is a right angle, the line KN is a diameter of the nine-point circle. $KP=\frac{1}{2}BO=BN$. Since KP and BN are equal and parallel, $KPBN$ is a parallelogram, and consequently $KN=BP$. $\therefore\ SN=\frac{1}{2}BP$. But SN is the radius of the nine-point circle and BP is the radius of the circumscribing circle of the triangle. Hence, the radius of the nine-point circle is half the radius of the circumscribing circle. $Q.\ E.\ D.$

(4.) Draw the medial lines BK, AG, and CH. Draw the line KH. Now the triangles KPH and BOC are similar, be-

cause the sides of the one are respectively parallel to the sides of the other, and the line HK is half the line BC, because H and K are the middle points of the sides AB and AC. $\therefore BO=2KP$. The triangles KPQ and BOQ are similar, because the angles of one are respectively equal to the angles of the other. Then we have $KP:KQ::BO:BQ$ or $KP:BO::KQ:BQ$. But $BO=2KP$. $\therefore BQ=2KQ$. $\therefore Q$ is the centroid and divides the line joining orthocenter and the center of the circumscribing circle in the ratio of $2:1$. $Q.E.D.$

Hence the line joining the centers of the circumscribing and nine-point circles is divided harmonically in the ratio of $2:1$ by the centroid and orthocenter of the triangles. These two points are therefore centers of similitude of the circumscribing and nine-point circles. $\therefore$ Any line drawn through either of these points is divided by the circumferences in the ratio of $2:1$.

(5.) Produce FE till it meets BC in P'. Since two opposite angles of the quadrilateral $BEFC$ are equal to two right angles, a circle may be circumscribed about it. Then we have $P'E.P'F=P'B.P'C$; therefore the tangents from P' to the circles are equal. $Q.E.D.$

(6.) Let O be the orthocenter, and I and Q the centers of the inscribed and circumscribed circles. Produce AI to bisect the arc BC in T. Bisect AO in L, and join GL, cutting AT in S. The nine-point circle passes through G, D, and L, and D is a right angle. Hence, GK is a diameter, and is therefore $=R=QA$. Therefore GL and QA are parallel. But $QA=QT$, therefore $GS=GT=CT\sin\frac{1}{2}A=2R\sin^2\frac{1}{2}A$. Also $ST=2GS\cos\theta$, θ being the angle $GST=GTS$. N being the center of the nine-point circle, its radius $=NG=\frac{1}{2}R$; and r being the radius of the inscribed circle, it is required to show that $NI=NG-r$. $NI^2=SN^2+SI^2-2SN.SI\cos\theta$. Substitute $SN=\frac{1}{2}R-GS$; $SI=TI-ST=2R\sin\frac{1}{2}A-2GS\cos\theta$; and $GS=2R\sin^2\frac{1}{2}A$, to prove the proposition. If J be the center of the escribed circle touching BC, r_1 its radius, it is shown in a similar way that $NJ=NG+r_1$.

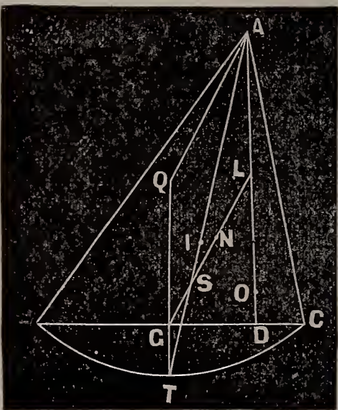


FIG. 7.

PROPOSITIONS.

1. The lines which join the middle points of adjacent sides of any quadrilateral, form a parallelogram.

2. Two medians of a triangle are equal; prove (without assuming that they trisect each other) that the triangle is isosceles.

3. In an indefinite straight line AB find a point equally distant from two given points which are not *both* on AB.
When does this problem not admit of solution?

Construct a right triangle having given:

4. The hypotenuse and the difference of the sides.

5. The perimeter and an acute angle.

6. The difference of the sides and an acute angle.

7. Construct a triangle having given the medians.

8. Construct a triangle, having given the base, the vertical angle, and(1) the sum or (2) the difference of the sides.

9. Describe a circle which shall touch a given circle at a given point, and also touch a given straight line.

10. Describe a circle which shall pass through two given points and be tangent to a given line.

11. Find the point inside a given triangle at which the sides subtend equal angles.

12. Describe a circle which shall be tangent to two intersecting straight lines and passing through a given point.

13. Divide a triangle in two equal parts by a line perpendicular to a side.

14. Inscribe in a triangle, a rectangle similar to a given rectangle.

15. Construct an equilateral triangle equivalent to a given square.

16. Trisect a triangle by straight lines drawn from a given point in one of its sides.

17. Draw through a given point a straight line, so that the part of it intercepted between a given straight line and a given circle may be divided at the given point in a given ratio.

18. Construct a circle equivalent to the sum of three given circles.

19, Find the locus of a point such that the sum of its distances from three given planes is equal to a given straight line.

20. Construct a sphere tangent to three given spheres and passing througn a given point.

21. Draw a circle tangent to three given circles.

Note.—This proposition is known as the *Taction Problem*. It was proposed and solved by Apollonius, of Pergæ, A. D. 200. His solution was indirect, reducing the problem to ever simpler and simpler problems. It was lost for centuries, but was restored by Vieta. The first direct solution was given by Gergonne, 1813. An elegant solution to this problem is given by Prof. E. B. Seitz, *School Visitor*, *Vol. IV*, *p. 61.*

22. Construct a sphere tangent to four given spheres.

Note.—This problem was first solved by Fermat (1601—1665).

23. The perpendicular from the center of gravity of a tetrahedron to any plane without the tetrahedron is one-fourth of the sum of the perpendiculars from the vertices to the same plane.

PROBABILITY PROBLEMS.

I. If three pennies be piled up at random on a horizontal plane, what is the probability that the pile will not fall down?

The pile will stand if the common center of gravity of the second and third coins falls on the surface of the first or bottom coin.

Let r be the radius of a penny; then the center of the second coin may fall anywhere in a circle whose radius is $2r$ and center the center of the surface of the first or bottom coin, and the center of the third coin may fall anywhere in a circle whose radius is $2r$ and the center the center of the surface of the second coin. The number of positions of the center of the second coin is therefore proportional to $4\pi r^2$, and for every one of these positions the center of the third coin can have $4\pi r^2$; hence the total number of positions of the second and third coins is proportional to $16\pi^2 r^4$.

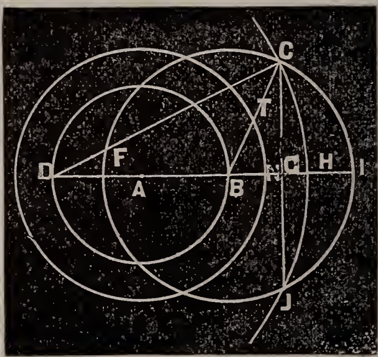


FIG. 8.

We must now determine in how many of these $16\pi^2 r^4$ positions the pile will stand.

Let A be the center of the first or bottom coin, and B the center of the second coin. Take $AD=AB$, and with center D and radius $2r$ describe the arc CHJ. If the center of the third coin is on the surface $CFJH$, the second and third coins will remain on the first, since $BN=NH$, $BT=TC$, and the pile will not fall down.

When AB is not greater than $\frac{1}{2}r$, the circle CHJ will not cut the surface of the second coin, and the pile will stand if the center of the third coin is anywhere on the second.

Let $AB=AD=x$, $S=$surface $CFJH$, and $p=$the probability required; then $DB=2x$, $BG=\dfrac{3r^2-4x^2}{4x}$, $DG=\dfrac{3r^2+4x^2}{4x}$,

$CG=\dfrac{1}{4x}\left[16r^4-(5r^2-4x^2)^2\right]^{\frac{1}{2}}$, arc $CI=r\cos^{-1}\left(\dfrac{3r^2-4x^2}{4rx}\right)$,

$$\text{arc } CH=2r\cos^{-1}\left(\frac{3r^2+4x^2}{8rx}\right),\ S=\pi r^2+4r^2\cos^{-1}\left(\frac{3r^2+4x^2}{8rx}\right)$$

$$-r^2\cos^{-1}\left(\frac{3r^2-4x^2}{4rx}\right)-\tfrac{1}{2}\Big[16r^4-(5r^2-4x^2)^2\Big]^{\frac{1}{2}},\ \text{and}$$

$$p=\frac{1}{16\pi^2r^4}\int_0^{\frac{1}{2}r}\pi r^2.2\pi xdx+\frac{1}{16\pi^2r^4}\int_{\frac{1}{2}r}^{r}S.2\pi xdx,$$

$$=\frac{1}{16}+\frac{1}{8\pi r^4}\int_{\frac{1}{2}r}^{r}(S-\pi r^2)xdx. \qquad \int r^2\cos^{-1}\left(\frac{3r^2-4x^2}{4rx}\right)xdx,$$

$$=\tfrac{1}{2}r^2x^2\cos^{-1}\left(\frac{3r^2-4x^2}{4rx}\right)-\tfrac{1}{2}r^4\cos^{-1}\left(\frac{5r^2-4x^2}{4r^2}\right)$$

$$+\frac{1}{16}r^2\Big[16r^4-(5r^2-4x^2)^2\Big]^{\frac{1}{2}}, \qquad \int 4r^2\cos^{-1}\left(\frac{3r^2+4x^2}{8rx}\right)xdx$$

$$=2r^2x^2\cos^{-1}\left(\frac{3r^2+4x^2}{8rx}\right)+\tfrac{1}{2}r^4\cos^{-1}\left(\frac{5r^2-4x^2}{4r^2}\right)$$

$$-\tfrac{1}{4}r^2\Big[16r^4-(5r^2-4x^2)^2\Big]^{\frac{1}{2}},\ \int\tfrac{1}{2}\Big[16r^4-(5r^2-4x^2)^2\Big]^{\frac{1}{2}}xdx$$

$$=\tfrac{1}{2}r^4\cos^{-1}\left(\frac{5r^2-4x^2}{4r^2}\right)-\frac{1}{32}(5r^2-4x^2)\Big[16r^4-(5r^2-4x^2)^2\Big]^{\frac{1}{2}}.$$

$$\therefore\ p=\frac{1}{16}-\frac{1}{8\pi r^4}\Big[\tfrac{1}{2}r^2x^2\cos^{-1}\left(\frac{3r^2-4x^2}{4rx}\right)-2r^2x^2\times$$

$$\cos^{-1}\left(\frac{3r^2+4x^2}{8rx}\right)-\tfrac{1}{2}r^4\cos^{-1}\left(\frac{5r^2-4x^2}{4r^2}\right)+\frac{1}{32}(5r^2-4x^2)\times$$

$$\sqrt{16r^4-(5r^2-4x^2)^2}\Big]_{\frac{1}{2}r}^{r}=\frac{1}{16}-\frac{3}{16\pi}\left(\frac{3}{16}\sqrt{15}-2\sin^{-1}\frac{1}{4}\right).$$

NOTE.—This solution is due to Artemas Martin, M. A., Ph. D., LL. D., member of the London Mathematical Society, member of the Edinburg Mathematical Society, member of the Mathematical Society of France, member of the New York Mathematical Society, member of the Philosophical Society of Washington and Fellow of the American Association for the Advancement of Science, Washington, D. C., who is one of the great peers of mathematical science.

BIOGRAPHY.

ARTEMAS MARTIN, M. A., PH. D., LL. D.

This eminent mathematician was born in Steuben county, N. Y., August 3, 1835. Early, his parents moved to Venango county, Pa., where they lived for many years. Dr. Martin had no schooling in his early boyhood, except a little primary instruction; but by self-application and indefatigable energy which have told the story of many a great man, he has become familiar to every mathematician and lover of science in every civilized country of the world.

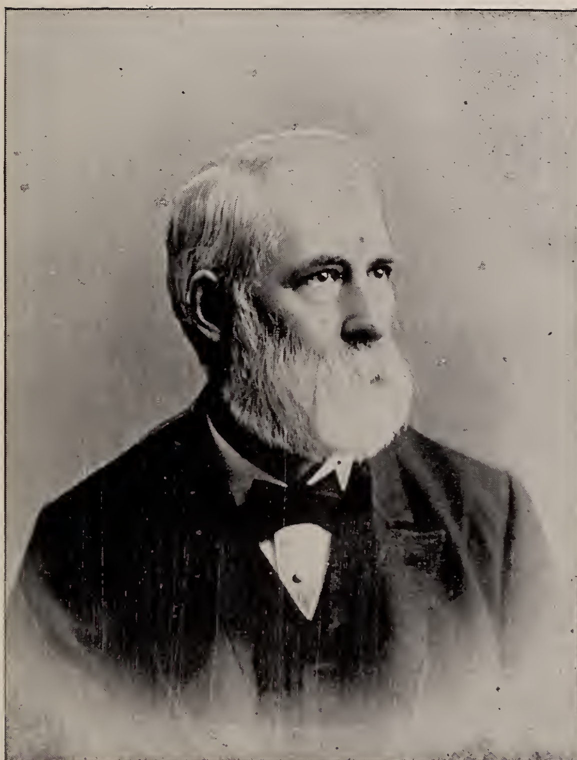
He was never a pupil at school, except when quite small, until in his fourteenth year. He had learned to read and write at home, but knew nothing of Arithmetic. At fourteen he commenced the study of Arithmetic, and after spending two winters in the district school, he commenced the study of Algebra. At seventeen, he studied Algebra, Geometry, Natural Philosophy, and Chemistry in the Franklin Select School, walking two and one-half miles night and morning. Three years after, he spent two and one-half months in the Franklin Academy, studying Algebra and Trigonometry. This finished his schooling. He taught district schools four winters, but not in succession. He was raised on a farm, and worked at farming and gardening in the summer; chopped wood in the winter; and after the discovery of oil in Venango county, worked at drilling oil wells a part of his time, always devoting his "spare moments" to study.

In the spring of 1869, the family moved to Erie county, Pa., where he resided until he entered the U. S. Coast Survey Office in 1885. While in Erie county, after 1871, he was engaged in market-gardening, which he carried on with great care and skill. He began his mathematical career when in his eighteenth year, by contributing solutions to the *Pittsburg Almanac*, soon after contributing problems to the "Riddler Column" of the *Philadelphia Saturday Evening Post*, and was one of the leading contributors for twenty years.

In the summer of 1864 he commenced contributing problems and solutions to *Clark's School Visitor*, afterward the *Schoolday Magazine*, published in Philadelphia. In June, 1870, he took charge of the "Stairway Department" as editor, the mathematical department of which he had conducted for some years before. He continued in charge as mathematial editor till the magazine was sold to Scribner & Co., in the spring of 1875, at which time it was merged into "*St. Nicholas*."

In September, 1875, he was chosen editor of a department of higher mathematics in the *Normal Monthly*, published by Prof. Edward Brooks, Millersville, Pa., and held that position till the *Monthly* was discontinued in August, 1876. He published in the *Normal Monthly* a series of sixteen articles on the Diophantine Analysis.

In June, 1877, Yale College conferred on him the honorary degree of Master of Arts (M. A.) In April, 1878, he was elected member of the London Mathematical Society. In June, 1882, Rutgers College conferred on him the honorary degree of Doctor of Philosophy (Ph. D.) March 7, 1884, he was elected a member of the Mathematical Society of France. In April, 1885, he was elected a member of the Edinburgh Mathematical Society. June 10, 1885, Hillsdale College conferred on him the honorary degree of Doctor of Laws (LL. D.) February 27, 1886, he was elected a member of the Philosophical Society of Washington. In June, 1881, he was elected Professor of Mathematics of the Normal School at Warrensburg, Mo., but did not accept the position. November 14, 1885, Dr. Martin was appointed



Yours truly
Artemas Martin

Librarian in the office of the U. S. Coast and Geodetic Survey. On August 26, 1890, he was elected a Fellow of the American Association for the Advancement of Science. On April 3, 1891 he was elected a member of the New York Mathematical Society.

All these honors have been worthily bestowed and the Colleges and Societies conferring them have done honor to themselves in recognizing the merits of one who has become such a power in the scientific world through his own efforts.

He has contributed fine problems and solutions to the following journals of the United States: *School Visitor*, *Analyst*, *Annals of Mathematics*, *Mathematical Monthly*, *Illinois Teacher*, *Iowa Instructor*, *National Educator*, *Yates County Chronicle*, *Barnes' Educational Monthly*, *Wittenberger*, *Maine Farmers' Almanac*, *Mathematical Messenger*, and *Educational Notes and Queries*. Besides other contributions, he contributed thirteen articles on "Average" to the Mathematical Department of the *Wittenberger*, edited by Prof. William Hoover. These are believed to be the first articles published on that subject in America.

Dr. Martin has also contributed to the following English mathematical periodicals: *Lady's*, and *Gentleman's Diary*, *Messenser of Mathematics*, and *The Educational Times and Reprint.*

The *Reprint* contains a large number of his solutions to difficult "Average" and "Probability" problems, which are master-pieces of mathematical thought and skill, and they will be lasting monuments to his memory. His style is direct, clear and elegant. His solutions are neat, accurate and simple. He has that rare faculty of presenting his solution in the simplest mathematical language, so that those who have mastered the elements of the various branches of mathematics, are able to understand his reasoning.

Dr. Martin is now (1893) editor of the *Mathematical Messenger*, and *The Mathematical Visitor*, two of the best mathematical periodicals published in America. These are handsomely arranged and profusely illustrated with very beautiful diagrams to the solutions, he doing the typesetting with his own hand. The typographical work of these journals is said to be the finest in America. The best mathematicians from all over the world contribute to these two journals. *The Mathematical Visitor* is devoted to Higher Mathematics, while *The Mathematical Magazine* is devoted to the solutions of problems of a more elementary nature. All solutions sent to Dr. Martin receive due credit, and if it is possible to find room for them the solutions are all published. He has thus encouraged many young aspirants to higher fields of mental activity. He is always ready to aid any one who is laboring to bring success with his work. He is of a kind and noble disposition and his generous nature is in full sympathy with every diligent student who is rising to planes of honor and distinction by self application and against adverse circumstances.

Dr. Martin has a large and valuable mathematical library containing many rare and interesting works. His collection of American arithmetics and algebras is one of the largest private collections of the kind in this country.

I. Find the average or mean distance of every point of a square from one corner.

Taking the corner from which the mean distance is to be found for the origin of orthogonal co-ordinates, and one of the sides of the square for the axis of abscissa, we have for the element of the surface $dx\ \ dy$, and since this element is at a distance $\sqrt{(x^2+y^2)}$ from the origin, the average $\frac{1}{a^2}\int_0^a\int_0^a dxdy\sqrt{(x^2+y^2)}$

$=\frac{1}{2a^2}\left\{a\int_0^a dx\sqrt{(a^2+x^2)}+\int_0^a x^2dx\log_e\frac{a+\sqrt{(a^2+x^2)}}{x}\right\}$. But

$\int x^2dx\log_e\frac{a+\sqrt{(a^2+x^2)}}{x}=\frac{1}{3}x^3\log_e\frac{a+\sqrt{(a^2+x^2}}{x}+\frac{1}{3}a\int\frac{x^2dx}{\sqrt{(a^2+x^2)}}$

$=\frac{1}{3}x^3\log_e\frac{a+\sqrt{(a^2+x^2)}}{x}+\frac{1}{6}ax\sqrt{(a^2+x^2)}-\frac{1}{6}a^3\log_e\{x+\sqrt{(a^2+x^2)}\}$

$\therefore$ Average$=\frac{1}{3}a[\sqrt{2}+\log_e(1+\sqrt{2})]$.

NOTE.—This solution is by Prof. J. W. F. Sheffer, Hagerstown, Md., whose name may be found attached to the solutions of many difficult problems proposed in the leading mathematical journals of the United States. The above solution is taken from the *Mathematical Messenger*, published by G. H. Harville, Simsboro, La.

I. All that is known concerning the veracities of two witnesses, A and B, is that B's statements are twice as reliable as A's. What is the probability of the truth of the concurrent testimony of these two witnesses?

Let x=the probability of the truth of any one of A's statements; then $2x$=the probability of any one of B's statements. The event did occur if both witnesses tell the truth, the probability of which is $x\times 2x=2x^2$. The event did not occur if both testify falsely, the probability of which is $(1-x)\times(1-2x)=1-3x+2x^2$. Hence, the probability of the occurrence of the event on the supposition that x is known is

$$p'=\frac{2x^2}{2x^2+(1-x)(1-2x)}.$$

Now, as the veracity of B can not exceed unity, the greatest value of x is found by putting $2x=1$, which gives $x=\frac{1}{2}$; hence, x can have any value from 0 to $\frac{1}{2}$.

Therefore, the probability in the problem is

$$p=\int_0^{\frac{1}{2}}p'dx\div\int_0^{\frac{1}{2}}dx=4\int_0^{\frac{1}{2}}\frac{x^2dx}{2x^2+(1-x)(1-2x)}$$

$$=64\int_0^{\frac{1}{2}}\frac{x^2dx}{(8x-3)^2+7}.$$

Let $8x-3=y$. Then $x=\frac{1}{8}(y\mid 3)$, $dx-\frac{1}{8}dy$; the limits of y are 1 and—3, and

$$p=\frac{1}{8}\int_{-3}^{+1}\frac{(y+3)^2dy}{y^2+7}=\frac{1}{8}\int_{-3}^{+1}\left(1+\frac{6y}{y^2+7}+\frac{2}{y^2+7}\right)dy$$

$$=\left[\frac{1}{8}y+\frac{3}{8}\log_e(y^2+7)+\frac{1}{4\sqrt{7}}\tan^{-1}\left(\frac{y}{\sqrt{7}}\right)\right]_{-3}^{+1}$$

$$=\tfrac{1}{2}-\tfrac{3}{8}\log_e 2+\frac{1}{4\sqrt{7}}\tan^{-1}\sqrt{7}.$$

NOTE.—This solution is taken from the *Mathematical Magazine, Vol. II, p. 122.* The solution there given is credited to the author, Prof. William Hoover, and Prof. P. H. Philbrick.

I. A cube is thrown into the air and a random shot fired through it; find the chance that shot passed through opposite faces.

Let AH be the cube. Through P, a point in the face $EFGH$, draw MK parallel to HE, and draw PN perpendicular to HE. Now if PA represents the direction of the shot, it will pass through the face $ABCD$, if it strikes the face $EFGH$ anywhere within $HMPN$.

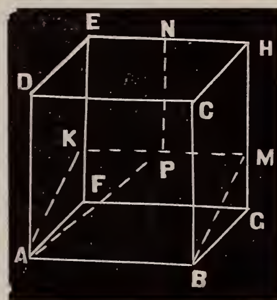


FIG. 9.

Let $AB=1$, $\angle KAF=\theta$, $\angle PAK=\phi$, and area $HMPN=u$. Then $AK=\sec\theta$, $PK=\sec\theta\tan\phi$, $FK=\tan\theta$, $AP=\sec\theta\sec\phi$, $PM=1-\sec'\tan\phi$, $PN=1-\tan\theta$, $u=(1-\sec\theta\tan\phi)(1-\tan\theta)$, the area of the projection of $HMPN$ on a plane perpendicular to $PA=u\cos\theta\cos\phi$, and that of $EFGH=\cos\theta\cos\phi$.

Since we are to consider all possible directions of the shot with respect to the cube, the points of intersection of PA with the surface of a sphere whose center is A, and radius unity, must be uniformly distributed. An element of the surface of this sphere is $\cos\phi\, d\theta\, d\phi$. By reason of the symmetry of the cube, the required chance is obtained by finding the number of ways the shot can pass through the opposite faces $EFGH$ and $ABCD$ between the limits $\theta=0$, and $\theta=\frac{1}{4}\pi$, and $\phi=0$ and $\phi=\tan^{-1}(\cos\theta)=\phi'$, and the number of ways it can pass through the face $EFGH$ between the limits $\theta=0$ and $\theta=\frac{1}{2}\pi$, and $\phi=0$ and $\phi=\frac{1}{2}\pi$; and then dividing the former by the latter. Hence, the chance required is

$$p=\frac{\int_0^{\frac14\pi}\int_0^{\Phi'} u\cos\theta\cos^2\phi\, d\theta\, d\phi}{\int_0^{\frac12\pi}\int_0^{\frac12\pi}\cos\theta\cos^2\phi\, d\theta\, d\phi}=\frac{4}{\pi}\int_0^{\frac14\pi}\int_0^{\Phi'} u\cos\theta\cos^2\phi\, d\theta\, d\phi,$$

$$=\frac{2}{\pi}\int_0^{\frac14\pi}(\cos\theta-\sin\theta)\tan^{-1}(\cos\theta)\,d\theta,$$

$$=\frac{2}{\pi}\Big[(\sin\theta+\cos\theta)\tan^{-1}(\cos\theta)-\theta+\sqrt{2}\tan^{-1}(\tfrac{1}{2}\sqrt{2}\tan\theta)$$

$$-\tfrac{1}{2}\log_e(1+\cos^2\theta)\Big]_0^{\frac14\pi}=\frac{1}{\pi}[4\sqrt{2}\tan^{-1}(\tfrac{1}{2}\sqrt{2})+\log_e(\tfrac{4}{3})-\pi].$$

NOTE.—This solution is due to Professor Enoch Beery Seitz, member of the London Mathematical Society, and late Professor of Mathematics in the North Missouri State Normal School, Kirksville, Mo.

BIOGRAPHY.

PROF. E. B. SEITZ, M. L. M. S.

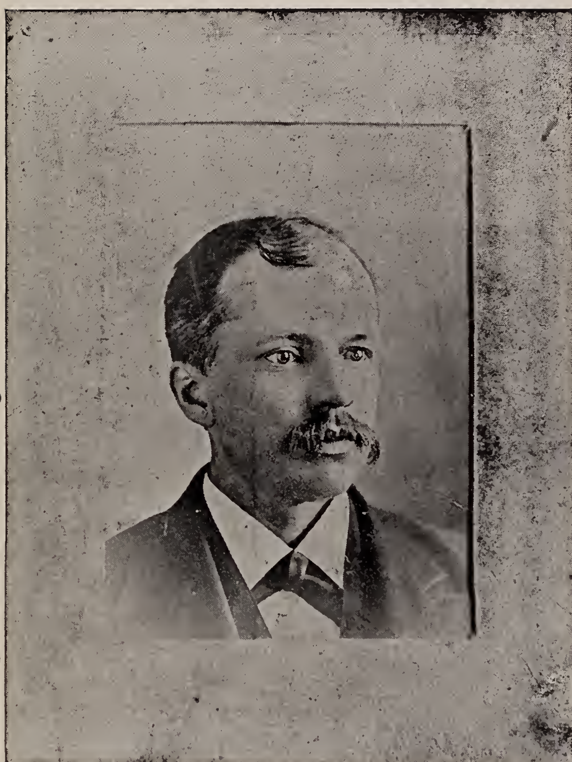
Professor Seitz, the most distinguished mathematician of his day, was born in Fairfield Co., O., Aug. 24, 1846, and died at Kirksville, Mo., Oct. 8, 1883. His father, Daniel Seitz was born in Rockingham Co., Va., Dec. 17, 1791, and was twice married. His first wife's name was Elizabeth Hite, of Fairfield Co., O., by whom he had eleven children. His second wife's name was Catharine Beery, born in the same county, Apr. 11, 1808, whom he married Apr. 15, 1832. This woman was blessed by four sons and three daughters. Mr. Seitz followed the occupation of a farmer and was an industrious and substantial citizen. He died near Lancaster, O., Oct. 14, 1864, in his seventy-third year; having been a resident of Fairfield Co. for sixty-three years.

Professor Seitz, the third son by his father's second marriage, passed his boyhood on a farm, and like most men who have become noted, had only the advantages of a common school education. Possessing a great thirst for learning, he applied himself diligently to his books in private and became a very fine scholar in the English branches, especially excelling in arithmetic. It was told the author, by his nephew, Mr. Huddle, that when Professor Seitz was in the field with a team, he would solve problems while the horses rested. Often he would go to the house and get in the garret where he had a few algebras upon which he would satiate his intellectual appetite. This was very annoying to his father who did not see the future greatness of his son, and many and severe were the floggings he received for going to his favorite retreat to gain a victory over some difficult problem upon which he had been studying while following the plow. Though the way seemed obstructed, he completed algebra at the age of fifteen, without an instructor. He chose teaching as his profession which he followed with gratifying success until his death. He took a mathematical course in the Ohio Wesleyan University in 1870. In 1872, he was elected one of the teachers in the Greenville High School, which position he held till 1879. On the 24th of June, 1875, he married Miss Anna E. Kerlin, one of Darke county's most refined ladies. In 1879, he was elected to the chair of mathematics in the Missouri State Normal School, Kirksville, Mo., which position he held till death called him from the confines of earth, ere his star of fame had reached the zenith of its glory. He was stricken by that "demon of death," typhoid fever, and passed the mysterious shades, to be numbered with the silent majority, on the 8th of October, 1883. On the 11th of March, 1880, he was elected a member of the London Mathematical Society, being the fifth American so honored.

Professor Seitz was in mathematics what Demosthenes was in oratory; Shakespear in poetry; and Napoleon in war: the equal of the best, the peer of all the rest.

He began his mathematical career in 1872, by contributing solutions to the problems proposed in the "Stairway" department of the *Schoolday Magazine*, conducted by Artemas Martin. His masterly and original solutions to difficult Average and Probability problems, soonattracted universal attention among mathematicians. Dr. Martin being desirous to know what works he had treating on that difficult subject, was greatly surprised to learn that he had no works upon the subject, but had learned what he knew about that difficult department of mathematical science by studying the problems and solutions in the *Schoolday Magazine* Afterwards, he contributed to the *Analyst*, the *Mathematical Visitor*, the *Wathematical Magazine*, the *School Visitor*, and the *Educational Times*, of London, Eng.

In all of these journals, Professor Seitz was second to none, as his logical and classical solutions to Average and Probability problems,rising as so many



Ever yours,

E. B. Seitz.

monuments to his untiring patience and indomitable energy and perseverence will attest. His name first appeared as a contributor to the *Educational Times* in Vol. XVIII., of *Reprint*, 1873. From this time until his death the *Reprint* is adorned with some of the finest product of his mighty intellect.

On page 21, Vol. II., he has given the above solution. This problem had been proposed in 1864 by the great English mathematician, Prof. Woolhouse, who solved it with great labor. It was said by an eminent mathematician of that day that the task of writing out a copy of that solution was worth more than the book in which it was published.

No other mathematician seemed to have the courage to investigate the problem after Prof. Woolhouse gave his solution to the world, till Professor Seitz took it up and demonstrated it so elegantly in half a page of ordinary type, that he fairly astonished the mathematicians of both Europe and America. Prof. Woolhouse was the best English authority on Probabilities, even before Professor Seitz was born.

It was the solution of this problem that won for Professor Seitz the acknowledgement of his superior ability over any other man in the world.

In studying his solutions, one is struck with the simplicity to which he has reduced the solutions of some of the most intricate problems. When he had grasped a problem in its entirity, he had mastered all problems of that class. He would so vary the conditions in thinking of one special problem and in effecting a solution that he had generalized all similar cases, so exhaustive was his analyses. Behind the words he saw all the ideas represented These he translated into symbols, and then he handled the symbols with a facility that has never been surpassed.

What he might have accomplished in his maturer years no man may say; but at the age of thirty-seven he laid down his pen and gave to God from whence it came, the casement and the key of his mighty intellect, leaving his impress indelibly stamped upon the thinking and scientific world for all time.

He was a man of the most singularly blameless life; his disposition was amiable; his manner gentle and unobtrusive; and his decision, when circumstances demanded it, was prompt and firm as the rocks. He did nothing from impulse; he carefully considered his course and came to conclusions which his conscience approved; and when his decision was made, it was unalterable.

Professor Seitz was not only a mathematician, but he was eminently proficient in other branches of knowledge. His mind was cast in a gigantic mold. "Being devout in heart as well as great in intellect, 'signs and quantities were to him but symbols of God's eternal truth' and 'he looked up through nature up to nature's God.' Professor Seitz, in the very appropriate words of Dr. Peabody regarding Benjamin Pierce, Professor of Mathematics and Astronomy in Harvard University, 'saw things precisely as they are seen by the infinite mind. He held the scales and compasses with which eternal wisdom built the earth, and meted out the heavens. As a mathematician, he was adored with awe. As a man, he was a christian in the whole aim and tenor of life.'" No mathematician was so universally loved and honored by his contemporaries as Professor Seitz.

Professor Seitz did not gain his knowledge from books, for his library consisted of only a few books and periodicals. He gained such a profound insight in the subtle relations of numbers by close application with which he was particularly gifted. He was not a mathematical genius, that is, as usually understood, one who is born with mathematical powers fully developed. But he was a genius in that he was especially gifted with the power to concentrate his mind upon any subject he wished to investigate. This happy faculty of concentrating all his powers of mind upon one topic to the exclusion of all others, and viewing it from all sides, enabled him to proceed with certainty where others would become confused and disheartened. Thread by thread and step by step, he took up and followed out long lines of thought and arrived at correct conclusions. The darker and more

subtle the question appeared to the average mind, the more eagerly he investigated it. No conditions were so complicated as to discourage him. His logic was overwhelming.

He left a wife and four sons—one of whom has gone to join his father in the realms of eternal peace. His mother, now (1893) eighty-five years old, is still living and enjoying good health.

TABLE I.—*Functions of π and e.*

$\pi = 3{\cdot}1415926$	$\pi^{-1} = {\cdot}3183099$	$e = 2{\cdot}71828183$
$\pi^2 = 9{\cdot}8696044$	$\pi^{-2} = {\cdot}1013212$	$e^2 = 7{\cdot}38905611$
$\pi^3 = 31{\cdot}0062761$	$\pi^{-3} = {\cdot}0322515$	$e^{-1} = 0{\cdot}3678794$
$\sqrt{\pi} = 1{\cdot}7724539$	$200^g \div \pi = 63^g{\cdot}6619772$	$e^{-2} = 0{\cdot}1353353$
$\log_{10}\pi = 1{\cdot}4971499$	$180^\circ \div \pi = 57^\circ{\cdot}2957795$	$\log_{10}e = 0{\cdot}43429448$
$\log_e \pi = 0{\cdot}6679358$	$= 206264''{\cdot}8$	$\log_e 10 = 2{\cdot}30258509$

TABLE II.

No.	Square root.	Cube root.
2	1·4142136	1·2599210
3	1·7320508	1·4422496
4	2·0000000	1·5874011
5	2·2360680	1·7099759
6	2·4494897	1·8171206
7	2·6457513	1·9129312
8	2·8284271	2·0000000
9	3·0000000	2·0800837
10	3·1622777	2·1544347
11	3·3166248	2·2239801
12	3·4641016	2·2894286
13	3·6055513	2·3513347
14	3·7416574	2·4101422
15	3·8729833	2·4662121
16	4·0000000	2·5198421
17	4·1231056	2·5712816
18	4·2426407	2·6207414
19	4·3588989	2·6684016
20	4·4721360	2·7144177
21	4·5825757	2·7589243
22	4·6904158	2·8020393
23	4·7958315	2·8438670
24	4·8989795	2·8844991
25	5·0000000	2·9240177
26	5·0990195	2·9624960
27	5·1961524	3·0000000
28	5·2915026	3·0365889
29	5·3851648	3·0723168
30	5·4772256	3·1072325

TABLE III.

N.	$\log_{10} N$.	$\log_e N$.
2	·3010300	·69314718
3	·4771213	1·09861229
5	·6989700	1·60943791
7	·8450980	1·94591015
11	1·0413927	2·39789527
13	1·1139434	2·56494936
17	1·2304489	2·83321334
19	1·2787536	2·94443898
23	1·3617278	3·13549422
29	1·4623980	3·36729583
31	1·4913617	3·43398720
37	1·5682017	3·61091791
41	1·6127839	3·71357207
43	1·6334685	3·76120012
47	1·6720979	3·85014760
53	1·7242759	3·97029191
59	1·7708520	4·07753744
61	1·7853298	4·11087386
67	1·8260748	4·20469262
71	1·8512583	4·26267988
73	1·8633229	4·29045944
79	1·8976271	4·36944785
83	1·9190781	4·41884061
89	1·9493900	4·48863637
97	1·9867717	4·57471098
101	2·0043214	4·61512052
103	2·0128372	4·63472899
107	2·0293838	4·67282883
109	2·0374265	4·69134788

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